Equilibrium Analysis of the All-Pay Contest with Two Nonidentical Prizes: Complete Results[*]

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Abstract

This paper studies contests in which three or more players compete for two nonidentical prizes. The players have distinct constant marginal costs of performance or bid, which are commonly known. We show that the contests have a generically unique Nash equilibrium, and it is in mixed strategies. Moreover, we characterize the equilibrium payoffs and strategies in closed form. We also study how the equilibrium payoffs and strategies vary with the prizes. As an application, we numerically compute the optimal allocation of prizes that maximizes the total expected bid of asymmetric players.

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1 Introduction

Contests with asymmetric players and heterogeneous prizes are predominant. For example, students of various intellectual levels compete for different grades, athletes of different abilities compete for different medals, and employees with different experience compete for different promotion opportunities. If we rank the prizes in a contest from the highest value to the lowest, we obtain a nonincreasing sequence of prize values, to which we refer as the prize sequence. The prize sequences in these contests have different shapes. For instance, in the 2016 U.S. Open tennis tournament, the prize is \$3.5 million for the winner, \$1.75 million for a runner-up, and \$0.875 million for a semifinalist. A prize is roughly half of the value of the next higher prize. In contrast, the prizes in the golf tournaments do not have the same property. For example, in 2016 U.S. Open golf tournament, the prizes are \$1.8 million for the champion, \$1.1 million for the runner-up, and \$0.68 million for the third place.

The shape of the prize sequence is especially important if the players have different abilities. To see why, if the prize sequence is very concave, the difference between higher prizes is small relative to that between lower prizes, which leads to less competition among the players with stronger abilities. In contrast, if the prize sequence is very convex, the difference between lower prizes is small relative to that between higher prizes, which leads to less competition among the players with lower abilities.

In this paper, we consider a complete-information all-pay contest among players of distinct constant marginal costs and two prizes of distinct values. This is the simplest setup to introduce prize sequences of different concavity/convexity, measured in the ratio of the difference between the two prizes to the difference between the lower prize and zero. We show that the contest has a unique Nash equilibrium, and it is in mixed strategies. In addition, we provide a closed-form characterization of the equilibrium payoffs and strategies, and computer programs to numerically compute them.

This paper's contribution is threefold. First, it shows equilibrium uniqueness. The uniqueness is not obvious because multiple equilibria have been found in contests with identical players (e.g. Baye et al. (1996)). In contrast, Siegel (2010) constructs a unique Nash equilibrium in contests with identical prizes and general nonlinear cost functions. This paper shows that his method, with non-trivial modifications, also applies to contests with asymmetric players and two distinct prizes, and can be used to show the uniqueness of Nash equilibrium.

Second, this paper provides a closed-form characterization of equilibrium payoffs and strategies in contests with two prizes of arbitrary values. As a result, it unifies the existing equilibrium characterizations with specific prize sequences, and we can illustrate how the unique equilibrium changes from one type to another as the prizes change. In addition, Xiao (2016) illustrates in an example that a convex prize sequence can lead to an equilibrium in which a player mixes over a non-interval set of bids. As a result of our closed-form characterization, we provide a necessary and sufficient condition for this to happen.

Third, this paper can be used to test conjectures on variants of all-pay auctions and contests

as well as on their design questions. If there is significant heterogeneity among either players or prizes, it is typically difficult to characterize equilibria in these games. However, our closed-form characterization and computer programs can be used to test conjectures and determine what results to expect. Specifically, we numerically compute the optimal allocation of prizes that maximizes the total expected bid of three asymmetric players. We find that the resulting optimal prize sequence contains either a single prize or two equal prizes, which complements the existing results by examining all the marginal cost profiles in a simplex, including the extreme values of marginal costs that have been previously studied.

Literature There is a large literature on contests and, closely related, auctions. See Konrad (2009) for a comprehensive survey. This paper is closely related to auctions and contests with complete information. As in this paper, Nash equilibria in these setups are usually in mixed strategies. A variety of prize structures are studied. For example, there is a large literature on contests with a single prize (e.g., Baye et al. (1996), Che and Gale (1998)). Identical prizes are considered by Clark and Riis (1998) and Siegel (2009, 2010). Arithmetic prize sequences – with constant first-order differences – are studied by Bulow and Levin (2006) and González-Díaz and Siegel (2013).[1] Xiao (2016) considers geometric prize sequences, with a constant ratio of two consecutive prizes, and quadratic prize sequences, with constant second-order differences, where both sequences are convex.

The main difference of this paper from the above is that we consider both concave and convex prize sequences. Moreover, we consider how the concavity/convexity affects asymmetric players. Barut and Kovenock (1998) study arbitrary prize sequences in contests among identical players. This paper extends their setup to asymmetric players but restricts it to two distinct prizes. Our findings are different from theirs. We find a unique equilibrium in contrast to their multiple equilibria. In addition, the prize allocation affects the total expected bid in our setup while the total expected bid is independent of prize allocations in their setup.[2] Azmat and Möller (2009) also consider symmetric players in a study of competing contests. Sela (2012) studies sequential all-pay auctions with one object in each stage. Olszewski and Siegel (2016a) study heterogeneous prizes and asymmetric players in large contests where the numbers of prizes and players go to infinity. In contrast, this paper considers a similar contest but with a finite number of prizes.

There is a literature on contests with asymmetric information, in contrast to the complete information in this paper. For example, Rosen (1986) studies the role of convex prize sequences in single-elimination tournaments, in which the players' effort is not observable. Moldovanu and Sela (2001) study the optimal allocation of prizes for ex ante symmetric players. All-pay auctions between two ex ante asymmetric players are studied in various setups (e.g., Amann and Leininger (1996), Lizzeri and Persico (2000), Siegel (2014), and Rentschler and Turocy (2016)).

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[1] Bulow and Levin (2006) study labor markets in which firms compete for workers. Their model can be transformed into a contest with arithmetic prize sequences.

[2] More precisely, for a fixed budget of prize money, any prize sequence whose lowest prize is zero maximizes the total expected bid.

However, we cannot study the effects of different prize structures on asymmetric players in those setups because they have either a single prize or symmetric players. Parreiras and Rubinchik (2010, 2015) study all-pay auctions of multiple objects and multiple ex ante asymmetric players. In contrast to this paper, the equilibria in those auctions are in pure strategies.

The remainder of this paper is organized as follows. Section 2 introduces a contest model among three players. Section 3 characterizes the equilibrium payoffs, and Section 4 characterizes the equilibrium strategies. Section 5 generalizes the results to more than three players and studies the optimal prize allocation for asymmetric players.

2   Model

For simpler notation, Sections 2 to 4 focus on a contest with three players $1, 2, 3$. Then, Section 5 extends the results to more players. Each player i has a constant marginal cost of bid $c_i > 0$, and the marginal costs are distinct $0 < c_1 < c_2 < c_3$.[3] Therefore, a bid $s_i \geq 0$ incurs a cost of $c_i s_i$ to player i. Player 1 is the strongest because it costs him the least to achieve the same bid. The contest has two distinct prizes $v_1 > v_2 > 0$.[4] Let $\mathbf{c} = (c_1, c_2, c_3)$ be the cost sequence and $\mathbf{v} = (v_1, v_2)$ be the prize sequence. Then, a contest is characterized by $(\mathbf{c}, \mathbf{v})$. The game is of complete information, so $(\mathbf{c}, \mathbf{v})$ is commonly known. Let the first order differences of the prizes be $\Delta_1 = v_1 - v_2$ and $\Delta_2 = v_2 - v_3$, where $v_3 = 0$. Then, the prize sequence is convex if $\Delta_1 > \Delta_2$, linear if $\Delta_1 = \Delta_2$, and concave if $\Delta_1 < \Delta_2$. We use the ratio Δ_1/Δ_2 to measure the convexity of the prize sequence, and we say a sequence is more convex than another if the ratio is larger.

Each player i chooses a bid $s_i \geq 0$ simultaneously. The player with the highest bid receives the highest prize v_1; the player with the second-highest bid receives the second-highest prize v_2; and the others receive no prize. In the case of a tie, ranks are allocated randomly with equal probabilities to tying players. For example, suppose $s_1 = s_2 > s_3$, then with probability $1/2$, player 1 receives v_1 and player 2 receives v_2; and with probability $1/2$, player 2 receives v_1 and player 1 receives v_2. If $s_1 > s_2 = s_3$, player 2 receives v_2 with probability $1/2$, and player 3 receives v_2 with probability $1/2$. If player i wins prize v_k with bid s_i, his payoff is $v_k - c_i s_i$; if a player chooses bid $s_i \geq 0$ but wins no prize, his payoff is $-c_i s_i$. All players are risk neutral. We consider only Nash equilibrium throughout the paper.

3   Equilibrium Payoffs

We first introduce a sequence of definitions, and show in Proposition 1 that the equilibrium payoffs can be constructed using the definitions. After that, Proposition 2 characterizes equilibrium payoffs in closed form, and Corollary 1 discusses comparative statics of the equilibrium payoffs with respect to the prize sequence.

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[3] If some players have identical marginal costs, there may be multiple Nash equilibria, so our uniqueness result does not apply. See, for instance, Baye et al. (1996).

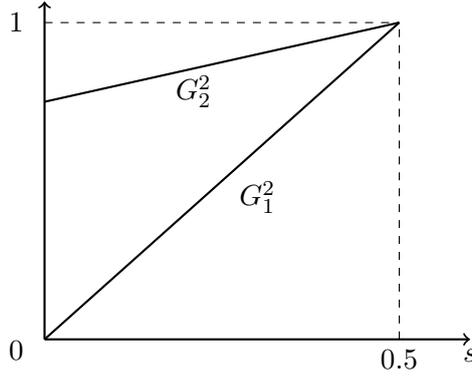[4] See Siegel (2010) for the case with $v_1 = v_2$.

Figure 1: A Two-Player Contest

We use a c.d.f. $G_i : [0, +\infty) \to [0, 1]$ to represent player i's (mixed) strategy. The support of G_i is the smallest closed set to which G_i assigns probability 1. Before the discussion of equilibrium payoffs, we introduce some notation in a two-player contest and a three-player contest.

First, consider a two-player contest in which the top two players 1 and 2 compete for prizes v_1 and v_2. The two-player contest is isomorphic to a two-player complete-information all-pay auction, and it is well-understood.[5] The contest has a unique equilibrium, and it is in mixed strategies.[6] The equilibrium strategies are

$$G_1^2(s) = c_2 s / (v_1 - v_2) \tag{1}$$

$$G_2^2(s) = 1 - c_1/c_2 + c_1 s / (v_1 - v_2) \tag{2}$$

for $s \in [0, \bar{s}_1^2]$, where $\bar{s}_1^2 = (v_1 - v_2)/c_2$. Throughout the paper, superscripts indicate the number of players in a contest. Figure 1 illustrates the equilibrium strategies if $c_1 = 1$, $c_2 = 4$ and $v_1 = 4$, $v_2 = 3$. The equilibrium payoffs are

$$u_i^2 = v_1 - (v_1 - v_2)c_i/c_2$$

for $i = 1, 2$.

Second, consider a three-player contest in which player 3 wins in every tie. This contest is the same as the original one described in Section 2 except the tie-breaking rule. More precisely, whenever player 3 has the same bid with another player, player 3 receives a higher prize than the other player.[7] We can extend G_i^2 for $i = 1, 2$ to a linear functions $G_i : \mathbb{R} \to \mathbb{R}$ such that $G_1(s) = c_2 s/(v_1 - v_2)$ and $G_2(s) = 1 - c_1/c_2 + c_1 s/(v_1 - v_2)$. Then, define a quadratic function $U_3(\cdot|G_1^2, G_2^2) : \mathbb{R} \to \mathbb{R}$ such that

$$U_3(s|G_1^2, G_2^2) \equiv v_1 G_1(s)G_2(s) + v_2[G_1(s)(1 - G_2(s)) + (1 - G_1(s))G_2(s)] - c_3 s \tag{3}$$

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[5]The contest is isomorphic to the complete-information all-pay auction with two players whose values are $(v_1 - v_2)/c_1$ and $(v_1 - v_2)/c_2$. The two games have the same equilibrium.

[6]See, for instance, Che and Gale (2006) and Kaplan and Wettstein (2006).

[7]Without this tie-breaking rule, the interpretation of $U_3(\cdot|G_1^2, G_2^2)$ provided below still applies for $s > 0$ but not for $s = 0$.

The function has an interpretation in the three-player contest in which player 3 wins in every tie. Specifically, if the other players 1 and 2 use strategies G_1^2 and G_2^2, player 3's expected payoffs from choosing s is $U_3(s|G_1^2, G_2^2)$, which is the expected value of his prizes $v_1 G_1^2(s)G_2^2(s) + v_2[G_1^2(s)(1 - G_2^2(s)) + (1 - G_1^2(s))G_2^2(s)]$ minus his cost $c_3 s$. Moreover, define

$$\hat{s}_3 \equiv \min \arg \max_{s \in [0, \bar{s}_1^2]} U_3(s|G_1^2, G_2^2) \tag{4}$$

which is player 3's smallest best response against G_1^2 and G_2^2. The minimum in (4) is necessary because multiple maximizers arise in two scenarios: First, if $v_1 - 2v_2 > 0$, the objective function $U_3(\cdot|G_1^2, G_2^2)$ is a U-shaped function, whose maximum over $[0, \bar{s}_1^2]$ may be reached at both boundaries of the interval. Second, if $v_1 - 2v_2 = 0$, $U_3(\cdot|G_1^2, G_2^2)$ reduces to a linear function, whose maximum over $[0, \bar{s}_1^2]$ may be reached at every point in the interval.

The expected payoff associated with the best response is

$$\hat{u}_3 \equiv U_3(\hat{s}_3|G_1^2, G_2^2) \tag{5}$$

and the corresponding expected value of prizes is

$$x_3 \equiv v_1 G_1^2(\hat{s}_3)G_2^2(\hat{s}_3) + v_2[G_1^2(\hat{s}_3)(1 - G_2^2(\hat{s}_3)) + (1 - G_1^2(\hat{s}_3))G_2^2(\hat{s}_3)] = \hat{u}_3 + c_3 \hat{s}_3 \tag{6}$$

Now we go back to the two-player contest, in which players 1 and 2 compete for v_1 and v_2, to define x_1 and x_2. Specifically, define

$$x_1 \equiv v_1 G_2^2(\hat{s}_3) + v_2(1 - G_2^2(\hat{s}_3)) \tag{7}$$

where $\hat{s}_3$ is defined as in (4). Given player 2's equilibrium strategy G_2^2 in this contest, x_1 can be interpreted as player 1's expected value of prizes from choosing $\hat{s}_3$.[8] Similarly, given player 1's strategy G_1^2, player 2's expected value of prizes from choosing $\hat{s}_3$ is

$$x_2 \equiv v_1 G_1^2(\hat{s}_3) + v_2(1 - G_1^2(\hat{s}_3)) \tag{8}$$

With this notation, we can introduce a sequence of definitions that are useful for characterizing equilibrium payoffs.

Definitions

i) Let G_1^2 and G_2^2 be the equilibrium strategies in the two-player contest in which players 1 and 2 compete for prizes v_1 and v_2. Consider the three-player contest in which player 3 wins in every tie. In this contest, define player 3's *value of winning* as $x_3 = U_3(\hat{s}_3|G_1^2, G_2^2) + c_3 \hat{s}_3$, which is his expected value of prizes at his smallest best response $\hat{s}_3$ against other players' strategies G_1^2 and G_2^2.

In the two-player contest with players 1 and 2 and prizes v_1 and v_2, for player $i = 1$ or

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[8]This interpretation applies for $\hat{s}_3 > 0$, but does not apply if $\hat{s}_3 = 0$ because G_2^2 has an atom at 0.

2, define his *value of winning* x_i as his expected value of prizes at $\hat{s}_3$ given the other player's equilibrium strategy G_j^2.

ii) The *threshold* T of the contest is the highest bid at which player 3's payoff is zero if his expected value of prizes is x_3, his value of winning.

iii) Player i's *power* w_i is his payoff at the threshold if his expected value of prizes is x_i, his value of winning.

The following example illustrates the above definitions.

Example 1 *Consider a contest of three players with marginal costs $c_1 = 1$, $c_2 = 4$, $c_3 = 7$ and prizes $v_1 = 4$, $v_2 = 3$. First, we derive the values of winning, x_1, x_2, x_3. Consider the two-player contest in which players 1 and 2 compete for prizes v_1 and v_2. The equilibrium strategies are $G_1^2(s) = 4s$ and $G_2^2(s) = 3/4 + s$.[9] Consider the three-player contest in which player 3 wins in every tie. Suppose players 1 and 2 use strategies G_1^2 and G_2^2, then player 3 has a unique best response $\hat{s}_3 = 1/8$, and the corresponding payoff is $\hat{u}_3 = 19/8$. According to Definition i), players 1 and 2's values of winning are defined in the two-player contest, and they are $x_1 = 31/8$ for player 1 and $x_2 = 7/2$ for player 2 according to (7) and (8). Player 3's value of winning is defined in the three-player contest in which player 3 wins in every tie, and it is $x_3 = \hat{u}_3 + c_3 \hat{s}_3 = 13/4$ according to (6). Then, following Definition ii), the threshold T satisfies $x_3 - c_3 T = 0$, so $T = 13/28$. Finally, according to Definition iii), player 1's power is $w_1 = x_1 - c_1 T = 191/56$, player 2's power is $w_2 = x_2 - c_2 T = 23/14$, and player 3's power is $w_3 = x_3 - c_3 T = 0$.*

In the contest in which 3 wins in every tie, we define the values of winning, which are the only modification we need to generalize the equilibrium payoff characterization of Siegel (2009) to our setup. To see this, if $v_1 = v_2 = v$, we have $G_1^2(\hat{s}_3) = G_2^2(\hat{s}_3) = 1$, so the value of winning in Definition i) is $x_i = v$ for all i, and the definitions of threshold and power are the same as those of Siegel. If $v_1 > v_2$, the definitions are different. In an equilibrium, a player may win v_1, v_2 and 0 with positive probability, so his expected value of prizes may be between v_1 and 0. The "value of winning" in Definition i) takes this into account. In Example 1, the values of winning are $x_1 = 31/8, x_2 = 29/8$ and $x_3 = 13/4$, all of which are between $v_1 = 4$ and 0. In addition, the value of x_i coincides with player i's expected value of prizes at the threshold T given other players' equilibrium strategies.

Using these definitions, the proposition below extends the payoff characterization of Siegel (2009) to contests with heterogeneous prizes.[10]

Proposition 1 *In every equilibrium of the contest, the expected payoff of every player equals the maximum of his power and 0.*

All proofs are relegated to the appendix.

To understand the idea behind the proof of Proposition 1, let an equilibrium (G_1^*, G_2^*, G_3^*) be given. Consider a contest with less competition by excluding player 3. In the resulting contest,

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[9]See the end of the appendix for the calculation details in Example 1.

[10]Siegel (2009) also considers very general nonlinear cost functions.

players 1 and 2 compete for v_1 and v_2. Because this new contest has less competition, if players 1 and 2 still use the equilibrium strategies G_1^* and G_2^*, their payoffs would be higher than those in the original equilibrium. Let $\bar{G}_1$ and $\bar{G}_2$ be strategies of players 1 and 2 such that their payoffs in the new contest remain the same as in the original equilibrium, then these strategies should be more "aggressive" than the strategies G_1^* and G_2^* in the original equilibrium.[11] A key step in the proof is that $\bar{s}_3^*$, the highest bid in player 3's equilibrium strategy's support, is a best response against $\bar{G}_1$ and $\bar{G}_2$ in the original three-player contest.[12] Intuitively, in the three-player contest, if we replace players 1 and 2's equilibrium strategies by the more aggressive strategies $\bar{G}_1$ and $\bar{G}_2$, no bid can give player 3 a payoff higher than his payoff in the original equilibrium. Moreover, it turns out that $\bar{s}_3^*$ can give player 3 his payoff in the original equilibrium, if 1 and 2's strategies are $\bar{G}_1$ and $\bar{G}_2$. This means that $\bar{s}_3^*$ is a best response of player 3 against $\bar{G}_1$ and $\bar{G}_2$ in the three-player contest.

Another property used in the proof is that G_1^2 and G_2^2, the equilibrium strategies in the two-player contest defined in Definition i), are $\bar{G}_1$ and $\bar{G}_2$ with a horizontal shift. Because of the horizontal shift, player 3 has the same expected value of prizes at his best response to $\bar{G}_1, \bar{G}_2$ and at his best response to G_1^2, G_2^2. Thus, player 3's value of winning in Definition i) is his expected value of prizes at $\bar{s}_3^*$ against others' equilibrium strategies in the three-player contest. This is also true for other players, i.e. x_i in Definition i) is i's expected value of prizes at $\bar{s}_3^*$ against others' equilibrium strategies. Therefore, to determine the equilibrium payoffs, we only need to determine $\bar{s}_3^*$. It is a property of any equilibrium that the payoff of 3, the weakest player, should be zero, so $\bar{s}_3^*$ must satisfy $x_3 - c_3 \bar{s}_3^* = 0$, which implies $\bar{s}_3^* = T$.

Proposition 1 transforms the problem of equilibrium payoff characterization – a fixed point problem – into a maximization problem of the quadratic function $U_3(\cdot|G_1^2, G_2^2)$. Therefore, we can use the proposition to find the equilibrium payoffs in closed form. In particular, because $U_3(\cdot|G_1^2, G_2^2)$ is a quadratic function, there are three possible cases for its maximum over the interval $[0, \bar{s}_1^2]$. In Case I, the lower boundary 0 is a maximizer, which may not be the unique maximizer. In Case II, the upper boundary $\bar{s}_1^2$ is the unique maximizer. In Case III, the maximizer is an interior point of $[0, \bar{s}_1^2]$. Case III arises if $U_3'(0|G_1^2, G_2^2) > 0$ and $U_3'(\bar{s}_1^2|G_1^2, G_2^2) < 0$, which are equivalent to

$$c_1 + c_2 < c_3 < 2c_1\Delta_2/\Delta_1 + c_2 - c_1 \tag{9}$$

If (9) does not hold, the maximizers are on the boundaries 0 and $\bar{s}_1^2$. Therefore, if (9) does not hold, Case II happens if $U_3(0|G_1^2, G_2^2) < U_3(\bar{s}_1^2|G_1^2, G_2^2)$, which is equivalent to

$$c_1/(c_3 - c_2) > \Delta_1/\Delta_2 \tag{10}$$

If neither (9) nor (10) holds, Case I arises.

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[11]This is formalized in Lemma 7.

[12]In Example 1, $\bar{s}_3^*$ is an interior point in the supports of $\bar{G}_1$ and $\bar{G}_2$. If $\bar{s}_3^*$ is at the lower boundary of the supports, we consider 3's best responses in the three-player contest in which 3 wins in every tie instead.
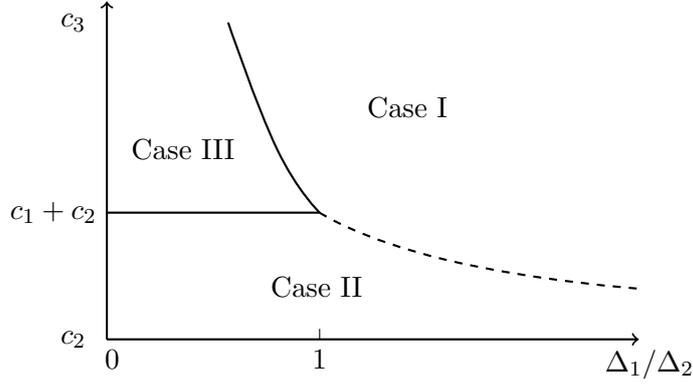
Figure 2: Equilibrium Types

Figure 2 illustrates these conditions for fixed c_1, c_2.[13] In the figure, Case I corresponds to the area with large c_3 and Δ_1/Δ_2. Recall that Δ_1/Δ_2 measures the prize sequence's convexity, so Case I arises with a very weak player 3 and a very convex prize sequence. Case II corresponds to the area with large Δ_1/Δ_2 but small c_3, which means a not too weak player 3 but a very convex prize sequence. Case III corresponds to large c_3 but small Δ_1/Δ_2, which means a very weak player 3 but not very convex prize sequence. The solid curve corresponds to the upper bound of c_3 in (9), and the dashed curve corresponds to the upper bound of Δ_1/Δ_2 in (10). Moreover, the two curves intersect at $(\Delta_1/\Delta_2, c_3) = (1, c_1 + c_2)$. The proposition below characterizes the equilibrium payoffs for each of the three cases.

Proposition 2 *In every equilibrium, $u_3^* = 0$.*
*(**Case I**) If neither (9) nor (10) holds, the equilibrium payoffs are*

$$u_1^* = v_1\left(1 - \frac{c_1}{c_2}\right) + v_2\left(\frac{c_1}{c_2} - \frac{c_1}{c_3} + \frac{c_1}{c_2}\frac{c_1}{c_3}\right), \ u_2^* = \frac{c_3 - c_2 + c_1}{c_3}v_2$$

*(**Case II**) If (9) does not hold but (10) does, the equilibrium payoffs are*

$$u_i^* = \left(1 - \frac{c_i}{c_3}\right)v_1 \ for \ i = 1, 2$$

*(**Case III**) If (9) holds, the equilibrium payoffs are*

$$u_1^* = v_1\left(1 - \frac{c_1}{c_2}\right) + v_2\frac{c_1}{c_2} - \frac{c_1}{c_3}\hat{u}_3, \ u_2^* = v_2 - \frac{c_2}{c_3}\hat{u}_3$$

where $\hat{u}_3$ is defined in (5) and

$$\hat{u}_3 = v_2\left(1 - \frac{c_1}{c_2}\right) - \frac{c_1 c_2}{4}(v_1 - 2v_2)\left[\frac{c_2 - c_1}{c_1 c_2} + \frac{c_1 + c_2}{c_1 c_2}\frac{v_2}{v_1 - 2v_2} - \frac{v_1 - v_2}{v_1 - 2v_2}\frac{c_3}{c_1 c_2}\right]^2 \quad (11)$$

We can verify that the equilibrium payoffs are continuous in the marginal costs and prizes.[14] The proposition above implies that the equilibrium payoffs are unique. Recall that we use

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[13]The curves are plotted for $c_1 = 1, c_2 = 2$.

[14]The conditions in Proposition 2 are sufficient but not necessary. For example, on the boundary between Case I and II in Figure 2, the equilibrium payoffs can be expressed as in Case II (and in Case I), but (10) is violated.

$(v_1 - v_2)/(v_2 - v_3)$, the ratio of first order differences, to measure the convexity of the prize sequence $\{v_1, v_2, v_3\}$ with $v_3 = 0$. Similarly, we use $(u_1^* - u_2^*)/(u_2^* - u_3^*)$ to measure the convexity of the payoff sequence $\{u_1^*, u_2^*, u_3^*\}$. The result below discusses the comparative statics of the convexity of the payoff sequence with respect to the convexity of the prize sequence.

Corollary 1 *Given* **c**, $(u_1^* - u_2^*)/(u_2^* - u_3^*)$ *is nondecreasing in* $(v_1 - v_2)/(v_2 - v_3)$. *That is, the sequence of equilibrium payoffs is weakly more convex if the prize sequence is more convex.*

4 Equilibrium Strategies

This section characterizes equilibrium strategies in closed form. We partition the parameter space of $(\mathbf{c}, \mathbf{v})$ into four subsets. Propositions 3-6 discuss the subsets separately. For each subset, the corresponding proposition characterizes a set of strategies in closed form and verifies that the strategies are indeed an equilibrium and it is the unique equilibrium. Based on the supports of the mixed strategies, we categorize the equilibrium into four cases: Case I, Case IIa, Case IIb, and Case III, which correspond to the four subsets in the partition of the parameter space. The equilibrium of Case I has payoffs as in Case I, the equilibrium of Case IIa or IIb has payoffs as in Case II, and the equilibrium of Case III has payoffs as in Case III.

To our knowledge, the Case III equilibrium has not been discussed in the literature. For example, Bulow and Levin (2006) illustrate the equilibrium of Case I or IIa if the prize sequence is arithmetic. Siegel (2010) shows the equilibrium of Case IIa if the prizes are identical. Xiao (2016) shows the equilibrium of Case IIb if the prize sequence is convex. By adding Case III, this paper provides a complete list of equilibrium cases.

Proposition 3 (Case I) *If* $\Delta_1/\Delta_2 \geq 1$ *and* $\Delta_1/\Delta_2 \geq c_1/(c_3 - c_2)$ *or if* $2c_1/(c_1 + c_3 - c_2) \leq \Delta_1/\Delta_2 \leq 1$, *the equilibrium payoffs are as in Case I, and the strategies in the unique equilibrium are*

$$G_1^*(s) = \frac{c_2 s + u_2^* - v_2}{v_1 - v_2} \quad for \ s \in \left[\frac{v_2 - u_2^*}{c_2}, \frac{v_1 - u_2^*}{c_2}\right]$$

$$G_2^*(s) = \begin{cases} \dfrac{c_3 s}{v_2} & for \ s \in \left[0, \dfrac{v_2 - u_2^*}{c_2}\right] \\[3mm] \dfrac{c_1 s + u_1^* - v_2}{v_1 - v_2} & for \ s \in \left[\dfrac{v_2 - u_2^*}{c_2}, \dfrac{v_1 - u_2^*}{c_2}\right] \end{cases}$$

$$G_3^*(s) = \frac{c_2 s + u_2^*}{v_2} \quad for \ s \in \left[0, \frac{v_2 - u_2^*}{c_2}\right]$$

The proofs of Propositions 3-6 are in the appendix. In each proof, we first show that an algorithm constructs a unique set of strategies and derive their closed-form characterization. In particular, we explain how to solve the strategies from an equation system and why the solution is unique. Then, we verify that the constructed strategies are indeed an equilibrium. After that, we show that every equilibrium must be one of the outcomes of the algorithm, which, combined with the unique outcome of the algorithm, implies equilibrium uniqueness.
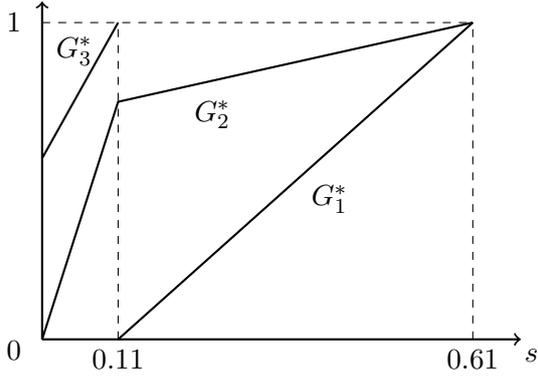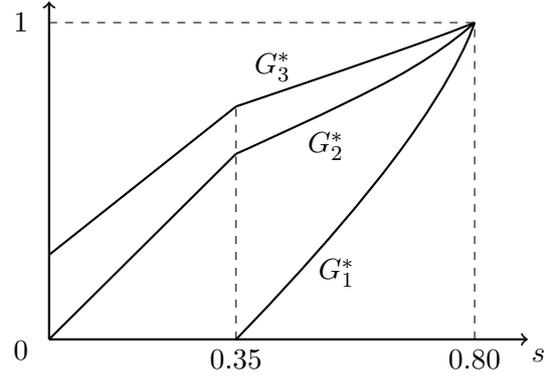
Figure 3: Case I Equilibrium

Figure 4: Case IIa Equilibrium

Intuitively, Case I arises if the first prize is much larger than the second, or if player 3's cost is much lower than 1 and 2's. As a result, the top two players, 1 and 2, compete for the first prize, while player 3 gives up the first prize and only compete for the second. More specifically, there are two intervals $[0, \Delta_2(c_2 - c_1)/(c_2 c_3)]$ and $[\Delta_2(c_2 - c_1)/(c_2 c_3), \Delta_2(c_2 - c_1)/(c_2 c_3) + \Delta_1/c_2]$. Player 1 mixes over the higher interval, player 3 mixes over the lower one, and player 2 mixes over both. For example, if $v_1 = 3, v_2 = 1$ and $c_1 = 1, c_2 = 4, c_3 = 7$, the contest has a Case I equilibrium. Figure 3 illustrates the equilibrium strategies.

Next, we discuss the Case IIa equilibrium. Define $A_i = u_i^* + (\Delta_2)^2/(\Delta_1 - \Delta_2)$ for $i = 1, 2, 3$, and $\underline{s}_1^*$ as the smallest $s \geq 0$ such that[15]

$$(v_1 - 2v_2)\frac{c_3 s}{v_2}\frac{u_2^* + c_2 s}{v_2} + v_2\left(\frac{c_3 s}{v_2} + \frac{u_2^* + c_2 s}{v_2}\right) - c_1 s = u_1^* \tag{12}$$

In addition, if $\Delta_1 \neq \Delta_2$, for $i = 1, 2, 3$ define

$$F_i(s) = \frac{1}{c_i s + A_i}\sqrt{\frac{\times_{j=1}^3 (c_j s + A_j)}{\Delta_1 - \Delta_2}} - \frac{\Delta_2}{\Delta_1 - \Delta_2} \tag{13}$$

which is repeatedly used in the characterization of the Case IIa, IIb and III equilibria.[16]

Proposition 4 (Case IIa) *If $\Delta_1/\Delta_2 > 1$, $\Delta_1/\Delta_2 < c_1/(c_3 - c_2)$ and*

$$\frac{\partial}{\partial s}\left(\frac{(A_1 + c_1 s)(A_2 + c_2 s)}{A_3 + c_3 s}\right)\bigg|_{\underline{s}_1^*} \geq 0 \tag{14}$$

or if $\Delta_1/\Delta_2 < 1$ and $c_3 \leq c_1 + c_2$, the equilibrium payoffs are as in Case II and the strategies

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[15]If $v_1 = 2v_2$, the equation below reduces to a linear equation, which may have a continuum of solutions.

[16]Under the conditions of Propositions 4-6, $F_i(s)$ is a real number. See, for example, the discussion below (28).

in the unique equilibrium are

$$G_1^*(s) = F_1(s) \quad \text{for } s \in [\underline{s}_1^*, v_1/c_3]$$

$$G_2^*(s) = \begin{cases} sc_3/v_2 & \text{for } s \in [0, \underline{s}_1^*] \\ F_2(s) & \text{for } s \in [\underline{s}_1^*, v_1/c_3] \end{cases}$$

$$G_3^*(s) = \begin{cases} sc_2/v_2 + (1 - c_2/c_3)\, v_1/v_2 & \text{for } s \in [0, \underline{s}_1^*] \\ F_3(s) & \text{for } s \in [\underline{s}_1^*, v_1/c_3] \end{cases}$$

If $\Delta_1/\Delta_2 = 1$ and $c_3 < c_1 + c_2$, the strategies are the same except that $G_i^*(s) = [\sum_{j=1}^3 (u_j^* + c_j s) - 2(u_i^* + c_i s)]/(2v_2)$ for $i = 1, 2, 3$ and for $s \in [\underline{s}_1^*, v_1/c_3]$.

Case IIa arises if the two prizes are similar or if the bottom two players, 2 and 3, have similar costs. Intuitively, the similarity in prizes and players leads to similarity in the highest bid. Specifically, in an equilibrium of Case IIa, all three strategies have interval supports, and their supports share the same upper boundary. For example, if $v_1 = 4, v_2 = 3$ and $c_1 = 2, c_2 = 4, c_3 = 5$, the contest has a Case IIa equilibrium. Figure 4 illustrates the equilibrium strategies.

Proposition 5 (Case III) *If $c_1 + c_2 < c_3 < c_2 - c_1 + 2c_2\Delta_2/\Delta_1$, the equilibrium payoffs are as in Case III, and the strategies in the unique equilibrium are*

$$G_1^*(s) = \begin{cases} F_1(s) & \text{for } s \in [\underline{s}_1^*, \bar{s}_3^*] \\ (c_2 s + u_2^* - v_2)/\Delta_1 & \text{for } s \in [\bar{s}_3^*, \bar{s}_1^*] \end{cases}$$

$$G_2^*(s) = \begin{cases} sc_3/v_2 & \text{for } s \in [0, \underline{s}_1^*] \\ F_2(s) & \text{for } s \in [\underline{s}_1^*, \bar{s}_3^*] \\ (sc_1 + u_1^* - v_2)/\Delta_1 & \text{for } s \in [\bar{s}_3^*, \bar{s}_1^*] \end{cases}$$

$$G_3^*(s) = \begin{cases} (sc_2 + u_2^*)/v_2 & \text{for } s \in [0, \underline{s}_1^*] \\ F_3(s) & \text{for } s \in [\underline{s}_1^*, \bar{s}_3^*] \end{cases}$$

where $\underline{s}_1^*$ is defined before Proposition 4, $\bar{s}_1^* = (v_1 - u_1^*)/c_1$ and $\bar{s}_3^* = T$.

Intuitively, this case arises if player 3's cost takes intermediate values, between $c_1 + c_2$ and $c_2 - c_1 + 2c_2\Delta_2/\Delta_1$, and if the prize sequence is not very convex, i.e., Δ_2/Δ_1 is bounded below. The three players compete in three bid intervals: over the high interval $[\bar{s}_3^*, \bar{s}_1^*]$, the top two players 1 and 2 compete for the first prize; over the low interval $[0, \underline{s}_1^*]$, the bottom two players 2 and 3 compete for the second prize, and over the intermediate interval $[\underline{s}_1^*, \bar{s}_3^*]$, the three players compete for both prizes. The following example illustrates an equilibrium of Case III.

Example 1 (continued) *Consider the contest in Example 1. The equilibrium payoffs are $u_1^* = 191/56$ for player 1, $u_2^* = 99/56$ for player 2, and $u_3^* = 0$ for player 3. All players' mixed strategies have interval supports. The supports are $[0, 13/28]$ for player 3, $[0, 33/56]$ for player 2, and $[(67 - 5\sqrt{37})/112, 33/56]$ for player 1. Figure 5 illustrates the equilibrium strategies. Two properties are worth mentioning: First, recall that the threshold of this contest is $T = 13/28$,*
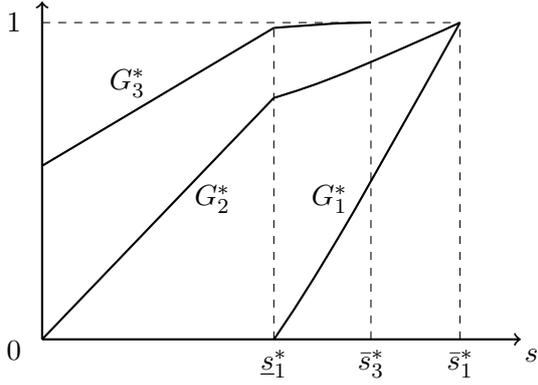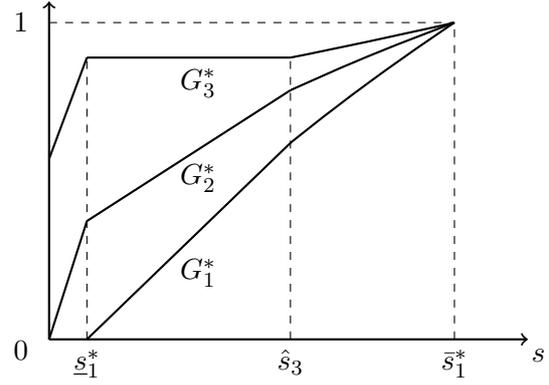
Figure 5: Case III Equilibrium

Figure 6: Case IIb Equilibrium

which is exactly the highest bid in the support of 3's equilibrium strategy. Second, we can verify that player i's "value of winning", x_i, equals the expected value of his prize at the threshold given the others' equilibrium strategies. Hence, player i's equilibrium payoff is $u_i^* = x_i - c_i T$.[17]

Proposition 6 (Case IIb) *If we have $\Delta_1/\Delta_2 > 1$, $\Delta_1/\Delta_2 < c_1/(c_3 - c_2)$ but not (14), the equilibrium payoffs are as in Case II, and the strategies in the unique equilibrium are*

$$G_1^*(s) = \begin{cases} \dfrac{c_2 s + u_2^* - v_2 G_3^*(\underline{s}_1^*)}{(\Delta_1 - \Delta_2) G_3^*(\underline{s}_1^*) + v_2} & \text{for } s \in [\underline{s}_1^*, \hat{s}_3] \\ F_1(s) & \text{for } s \in [\hat{s}_3, \bar{s}_1^*] \end{cases}$$

$$G_2^*(s) = \begin{cases} s c_3 / v_2 & \text{for } s \in [0, \underline{s}_1^*] \\ \dfrac{c_1 s + u_1^* - v_2 G_3^*(\underline{s}_1^*)}{(\Delta_1 - \Delta_2) G_3^*(\underline{s}_1^*) + v_2} & \text{for } s \in [\underline{s}_1^*, \hat{s}_3] \\ F_2(s) & \text{for } s \in [\hat{s}_3, \bar{s}_1^*] \end{cases}$$

$$G_3^*(s) = \begin{cases} s c_2 / v_2 + (1 - c_2/c_3) v_1 / v_2 & \text{for } s \in [0, \underline{s}_1^*] \\ \underline{s}_1^* c_2 / v_2 + (1 - c_2/c_3) v_1 / v_2 & \text{for } s \in [\underline{s}_1^*, \hat{s}_3] \\ F_3(s) & \text{for } s \in [\hat{s}_3, \bar{s}_1^*] \end{cases}$$

where $\hat{s}_3$ is the smallest $s \geq \underline{s}_1^*$ such that $U_3(s|F_1, F_2) = u_3^*$.

Intuitively, this case arises when the bottom two players 2 and 3 have similar costs, and if the prize sequence is convex. The three players compete in three bid intervals as in Case III. The difference is that, over the high interval $[\hat{s}_3^*, \bar{s}_1^*]$, the three players compete for both prizes, which is a result of players 2 and 3's similar costs. Proposition 6 provides a necessary and sufficient condition on the convexity of the prize sequence for equilibrium strategies with non-interval supports. This complements Siegel's (2010) result that, in contests with identical prizes, linear cost functions result in equilibrium strategies with interval supports, but nonlinear cost functions may not. The following example illustrates an equilibrium of Case IIb.

Example 2 *Consider a contest of three players with marginal costs $c_1 = 4$, $c_2 = 6$, $c_3 = 7$ and prizes $v_1 = 4$, $v_2 = 1$. The equilibrium payoffs are $u_1^* = 1.71$ for player 1, $u_2^* = 0.57$ for player*

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[17]Both properties are true in general. We can verify the properties using the closed-form characterization of equilibrium strategies.

2, *and zero for player* 3. *Player* 1 *mixes over interval* $[0.05, 0.57]$, *player* 2 *mixes over* $[0, 0.57]$, *and player* 3 *mixes over* $[0, 0.05] \cup [0.34, 0.57]$. *The equilibrium strategies are*[18]

$$
G_1^*(s) = \begin{cases} 2.16s - 0.11 & \text{for } s \in [0.05, 0.34] \\ \sqrt{0.5(84s+15)(14s+1)/(56s+31)} - 0.5 & \text{for } s \in [0.34, 0.57] \end{cases}
$$

$$
G_2^*(s) = \begin{cases} 7s & \text{for } s \in [0, 0.05] \\ 1.44s + 0.30 & \text{for } s \in [0.05, 0.34] \\ \sqrt{0.5(56s+31)(14s+1)/(84s+15)} - 0.5 & \text{for } s \in [0.34, 0.57] \end{cases}
$$

$$
G_3^*(s) = \begin{cases} 6s + 0.57 & \text{for } s \in [0, 0.05] \\ 0.89 & \text{for } s \in [0.05, 0.34] \\ \sqrt{0.5(84s+15)(56s+31)/(14s+1)} - 0.5 & \text{for } s \in [0.34, 0.57] \end{cases}
$$

Given G_1^* *and* G_2^*, *player* 3*'s payoff from choosing* $s \in (0.05, 0.34)$ *is a U-shaped quadratic curve passing* 0 *at the boundaries of the interval, so the payoff is lower than* 0. *Figure* 6 *illustrates the equilibrium strategies.*

Combining Propositions 3-6, we have the following result.

Corollary 2 *The contest has a unique Nash equilibrium.*

According to Propositions 3-6, the unique equilibrium can be one of four cases. These propositions unify the existing equilibrium characterizations for different specific prize sequences. In addition, we can illustrate how the different equilibrium cases relate to each other. More precisely, we consider how the unique equilibrium changes from one case to another as the prize sequence becomes more convex.

Proposition 7 *If* $c_3 > c_1 + c_2$, *the equilibrium is of Case IIa for* $\Delta_1/\Delta_2 = 0$; *Case III for* $\Delta_1/\Delta_2 \in (0, 2c_1/(c_1 + c_3 - c_2))$; *and Case I for* $\Delta_1/\Delta_2 > 2c_1/(c_1 + c_3 - c_2)$.
If $c_3 \leq c_1 + c_2$, *there exists* $\lambda \in (1, c_1/(c_3 - c_2))$ *such that the equilibrium is of Case IIa for* $\Delta_1/\Delta_2 < \lambda$, *Case IIb for* $\Delta_1/\Delta_2 \in (\lambda, c_1/(c_3 - c_2))$; *and Case I for* $\Delta_1/\Delta_2 \geq c_1/(c_3 - c_2)$.

The transition is demonstrated in Figure 7. In the figure, we use the supports of the equilibrium strategies to demonstrate different cases. If Δ_1/Δ_2 is small, the equilibrium is of Case II; if Δ_1/Δ_2 is large enough, the equilibrium is of Case I. How does the equilibrium transform from Case IIa to Case I as Δ_1/Δ_2 increases? If $c_3 > c_1 + c_2$, the equilibrium changes from Case IIa to Case IIb then to Case I, which is illustrated at the upper half of the figure. If $c_3 \leq c_1 + c_2$, the equilibrium changes from Case IIa to Case III then to Case I, which is illustrated at the lower half of the figure.

We show above the unique equilibrium in contests with linear cost functions and two distinct prizes. We describe below two potential extensions and what we know about them. First, the method in this paper can be extended to show equilibrium uniqueness if the cost functions are

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[18]The slopes and intercepts of the linear parts of the strategies are rounded to two decimal places.
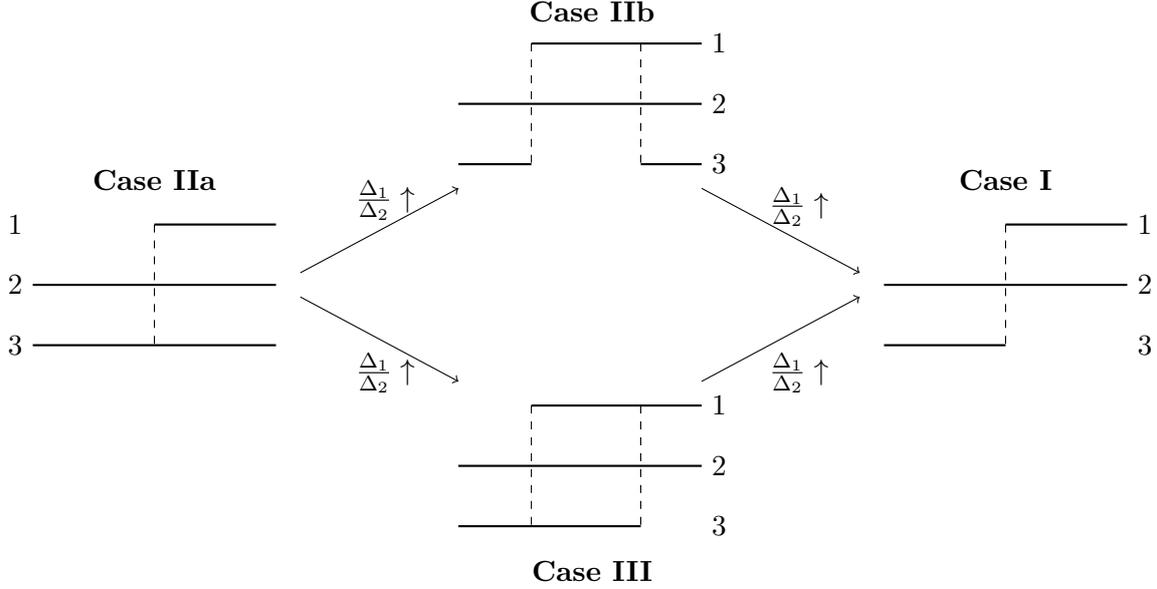
Figure 7: Supports of Mixed Strategies

nonlinear and have ordered marginal cost functions. However, the equilibrium generally has no closed-form characterization. Another extension is to consider more than two positive prizes. However, equilibrium strategies generally do not have a closed-form characterization either.[19]

5 Application and Extension

5.1 Revenue Maximizing Prizes

Consider a contest organizer with a fixed budget whose value is normalized to 1. She is risk neutral and wants to maximize the revenue – the total expected bid – by splitting the budget into v_1 and v_2 such that $v_1, v_2 \geq 0$ and $v_1 + v_2 = 1$. Because there are three players, it is not optimal to have three or more prizes. Because of the closed-form characterization of the equilibrium strategies, we can numerically calculate the expected bid given each prize allocation. Therefore, we can find the optimal prize allocation (v_1^*, v_2^*) given any cost profile $\mathbf{c} = (c_1, c_2, c_3)$.

In Figure 8, the triangle on the right illustrates the optimal prize allocation for $\mathbf{c} = (c_1, c_2, c_3)$ satisfies $c_1 + c_2 + c_3 = 1$ and $0 \leq c_1 \leq c_2 \leq c_3$. For a given c_1, the dashed line represents $\mathbf{c}$ such that $c_2 + c_3 = 1 - c_1$. Moreover, moving up along the dashed line leads to higher values of $(c_3 - c_2)/(c_2 - c_1)$. Therefore, we have two key observations from the figure:

- *The optimal allocation (v_1^*, v_2^*) is either $(1/2, 1/2)$ or $(1, 0)$.*

- *For a given c_1, there exists $\phi \in (0, \infty)$ such that the optimal allocation is $(v_1^*, v_2^*) = (1/2, 1/2)$ if $(c_3 - c_2)/(c_2 - c_1) < \phi$, and $(v_1^*, v_2^*) = (1, 0)$ if $(c_3 - c_2)/(c_2 - c_1) > \phi$.*

This means that a single prize – the most convex prize sequence – maximizes the total expected bid in a three-player contest if the top two players are similar, and two equal prizes – the most

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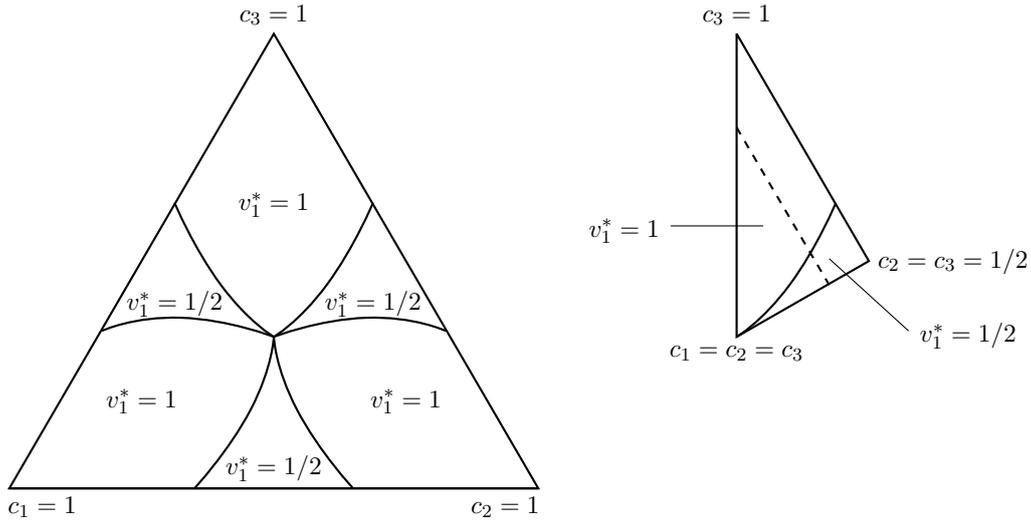[19]See Xiao (2016) for examples of such equilibria.

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Figure 8: Performance Maximizing Prizes

concave prize sequence – maximize the total expected bid if the bottom two players are similar. Moreover, according to Corollary 1, if the top two players are similar, the revenue maximizing prizes result in the most convex equilibrium payoff sequence; if the bottom two players are similar, the revenue maximizing prizes result in the least convex equilibrium payoff sequence. By renaming the players, the above results extend to the entire simplex in which $\mathbf{c} = (c_1, c_2, c_3)$ satisfies $c_1 + c_2 + c_3 = 1$ and $c_i \geq 0$ for $i = 1, 2, 3$. In Figure 8 the triangle on the left illustrates the optimal prize allocation over the entire simplex.[20]

There is a big literature on contests with multiple prizes (see Sisak (2009) for a survey), and multiple prizes are studied in various scenarios, e.g. contests with participation constraints (Megidish and Sela (2013)). Our analysis above focuses on how the asymmetry of players' costs affects the optimal allocation of prizes. In a different setup with ex ante symmetric players, Moldovanu and Sela (2001) study how the convexity of the players' cost function affects the optimal prize allocation. They show that a single prize is optimal if the players have concave or linear cost functions, and multiple prizes are optimal if they have concave cost functions. In contrast, our results suggest that in the case of complete information, multiple prizes can be optimal even if the cost functions are linear.

The optimality of multiple prizes is also demonstrated in various limiting cases. For example, Szymanski and Valletti (2005) and Xiao (2016) consider contests in which the strongest player's marginal cost converges to zero. Cohen and Sela (2008) study an all-pay auction in which one player values the second prize slightly higher than the other players. Our findings complement those results by examining all the cost profiles in the simplex, including the extreme values of marginal costs.

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[20]In a zero-measure subset of the simplex, $\mathbf{c}$ contains identical marginal costs. For those values of $\mathbf{c}$, our method still constructs an equilibrium, which we use in the simulation. However, there may be other equilibria.

5.2 More Than Three Players

Consider the same contest described in Section 2 except that there are more than three players: $1, 2, ..., n$, where $n \geq 3$. They have constant marginal costs of bid, which satisfy $0 < c_1^n < c_2^n < ... < c_n^n$. The following result extends the closed form characterization of equilibrium payoffs and strategies to the n-player contest.

Proposition 8 *The n-player contest has a unique equilibrium, in which players $1, 2, 3$'s payoffs and strategies are the same as in the three-player contest, the others choose $s = 0$ with probability 1 and receive a payoff of zero.*

Next, we consider the revenue maximizing prizes for $n > 3$ players. Olszewski and Siegel (2016b) examine revenue maximizing prizes in large contests in which the number of players goes to infinity. They find that one prize is optimal if the limiting distribution of the players' constant marginal costs has a continuous and strictly positive density.[21] Our finding is different from theirs. Specifically, for $n > 6$, consider an n-player contest with $c_1^n = 1/n, c_2^n = 1/2 - 2/n$ and $c_i^n = 1/2 + (i-2)/n$ for $i \geq 3$. Note that $0 < c_1^n < c_2^n < ... < c_n^n$ and $c_1^n + c_2^n + c_3^n = 1$. Given one or two prizes, players $4, ..., n$ choose zero bid with probability 1, so the total expected bid in this contest is the same as the three-player contest among players 1, 2 and 3. Notice that $\lim_{n \to +\infty} c_3^n - c_2^n = 0$, so our observations above suggest that, for a large enough n, two equal prizes result in higher expected bid than one prize in the n-player contests. The main reason for the difference is that, if $n \to +\infty$, the limiting distribution of the marginal costs is $F(c) = 0$ for $c \in [0, 1/2]$ and $F(c) = c - 1/2$ for $c \in [1/2, 3/2]$. Its density function over $(0, 1/2)$ is 0, which is excluded by Olszewski and Siegel (2016b).

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[21]See their Proposition 7.

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Appendix

We first introduce several known equilibrium properties in Lemmas 1 to 4, then some additional properties in Lemmas 5 to 7. See Step 2 in Appendix A of Bulow and Levin (2006) for Lemma 1 and Xiao (2016) for Lemmas 2-4. These papers consider different prize sequences, but their proofs apply here. After that, we use the equilibrium properties to prove Proposition 1.

It is easy to see that there is no equilibrium in pure strategies.[22] For an equilibrium with strategies $G_1^*, ..., G_n^*$ in the n-player contest, let $\underline{s}_i^*$ and $\bar{s}_i^*$ be the minimal and maximum bids in the support of G_i^*.

Lemma 1 *(No Aggregate Gaps)* *In every equilibrium, if $s \in [0, \max(\bar{s}_1^*, ..., \bar{s}_n^*)]$, there are at least two players whose equilibrium strategies' supports contain s.*

As a result, in any equilibrium, if s is in the support of one player's strategy, and s' in the support of another player's strategy, then any bid between s and s' is in the support of some player's strategy.

Lemma 2 *(Participation)* *In every equilibrium, player $i > 3$ assigns probability 1 to $s = 0$.*

The above lemma implies that the top three players, whose costs are the lowest, choose positive bids to compete for the two prizes. The other players' costs are too high and they give up by choosing $s = 0$.

Lemma 3 *(Nested Gaps)* *In every equilibrium, if $s \in (\underline{s}_i^*, \bar{s}_i^*)$ is not in the support of G_i^*, then s is not in the support of G_j^* for any $j > i$.*

If $s \in (\underline{s}_i^*, \bar{s}_i^*)$ is not in the support of G_i^*, it means that, due to the competition from other players, s is not a best response for player i. Then, Lemma 3 implies that for the players weaker than i, the bid s is not a best response either.

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[22]See, for instance, Bulow and Levin (2006).

Lemma 4 (Stochastic Dominance) *In every equilibrium, if $i < j$, then $G_i^*(s) \leq G_j^*(s)$ for $s \geq 0$.*

If $i < j$, player i is stronger than j because i has a lower marginal cost. Then, the lemma means the equilibrium bids of a stronger player are higher, in terms of the first order stochastic dominance, than those of a weaker player.

Lemma 5 *Suppose a player has an atom at bid s in an equilibrium, that is, he chooses s with positive probability. Then, he receives no prize by choosing s.*

Proof. We first show that, if two or more players have an atom at bid s in an equilibrium, all the players who have an atom at s lose with certainty.[23] Let us prove it by contradiction. Suppose that two players, i and j, have an atom at bid s in an equilibrium, and suppose that player i wins a prize with positive probability by choosing s. Since the tie is broken in such a way that everyone involved wins with positive probability, player j also wins a prize with positive probability by choosing s. In addition, the tie breaking rule ensures that player i loses with positive probability by choosing s, so he does not win the highest prize with probability 1. In contrast, if player j increases his bid slightly above s, his cost is almost the same but his expected winnings would have a discontinuous increase. This is because he no longer needs to share any prize with player i. This is a deviation for player j, which is a contradiction.

Next, using the above claim, we prove the lemma in two steps. First, suppose two players have an atom at bid s in the equilibrium, then the above claim implies that both of them must lose with certainty by choosing s. Second, suppose only player i has an atom at s, and suppose he wins a prize with positive probability. On the one hand, if all other players have no best response in $(s - \varepsilon, s)$ for some $\varepsilon > 0$, player i would benefit from lowering the atom to $s - \varepsilon$. This is a contradiction. On the other hand, suppose another player j has a sequence of best responses converging to s from below. Compared to such a best response close to s, a bid slightly above s imposes an almost identical cost on player j, but the resulting expected winnings would have a discontinuous increase because of player i's atom at s. This is also a contradiction. In sum, player i loses with certainty by choosing bid s, which completes the proof. ∎

According to the lemma, a player never has an atom at $s > 0$ in an equilibrium, otherwise he receives a negative expected payoff. Therefore, the possible atoms in an equilibrium must be at $s = 0$. The lemma below shows that only the weaker players, 3, 4, ..., n, bid zero with positive probability.

Lemma 6 *In every equilibrium, players 1 and 2's strategies G_1^* and G_2^* have no atoms; player 3's strategy G_3^* has an atom at $s = 0$; and player $i \geq 4$ assigns probability 1 to $s = 0$.*

Proof. We prove this in three steps. First, Lemma 2 implies that player $i \geq 4$ assigns probability 1 to $s = 0$. As a result, given others' equilibrium strategies, a player's payoff at a positive bid does not depend on the strategies of $i \geq 4$.

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[23]This claim is referred to as the Tie Lemma by Siegel (2009).

Second, G_1^* and G_2^* have no atoms. Suppose otherwise that in an equilibrium, G_1^* has an atom. Then, Lemma 5 implies that player 1's expected equilibrium payoff $u_1^* = 0$. Recall that $\bar{s}_3^*$ is the highest bid in the support of G_3^*. If $\bar{s}_3^* = 0$, player 1 could receive a positive expected payoff by deviating to $s = 0$. Therefore, $\bar{s}_3^* > 0$. Due to Lemma 5, $\bar{s}_3^* > 0$ cannot be an atom. Hence, given others' equilibrium strategies, player 1's expected payoff at $\bar{s}_3^*$ is $u_1^* = U_1(\bar{s}_3^*|G_2^*, G_3^*)$, where

$$U_i(s|G_j, G_k) = v_1 G_j(s)G_k(s) + v_2[G_j(s)(1 - G_k(s)) + (1 - G_j(s))G_k(s)] - c_i s$$

Notice that player 1's payoff at $\bar{s}_3^*$ is independent of G_i^* for $i \geq 4$, which is due to the first step. Similarly, $u_3^* = U_3(\bar{s}_3^*|G_2^*, G_1^*)$. In addition, because $G_3^*(\bar{s}_3^*) = 1 \geq G_1^*(\bar{s}_3^*)$ and $c_3 \bar{s}_3^* > c_1 \bar{s}_3^*$, we have $U_1(\bar{s}_3^*|G_2^*, G_3^*) > U_3(\bar{s}_3^*|G_2^*, G_1^*)$. As a result, $u_1^* > u_3^* \geq 0$, which contradicts $u_1^* = 0$.

Third, G_3^* has an atom at $s = 0$. Suppose otherwise that $G_3^*(0) = 0$. Recall that $\underline{s}_i^*$ is the smallest bid in the support of G_i^*. Then, if $\min(\underline{s}_1^*, \underline{s}_2^*) > 0$, the interval $(0, \min(\underline{s}_1^*, \underline{s}_2^*))$ receives zero probability from G_i^* for $i \neq 3$. This contradicts Lemma 1, the property of "No Aggregate Gaps". Therefore, $\min(\underline{s}_1^*, \underline{s}_2^*) = 0$. Without loss of generality, assume $\underline{s}_2^* = 0$. Recall that $G_1^*(0) = 0$ according to the second step. Then, the assumption $G_3^*(0) = 0$ implies that, given others' equilibrium strategies, player 2's expected payoff at $\underline{s}_2^* = 0$ is zero. Therefore, $u_2^* = 0$, which leads to a contradiction by the same argument in the second step. Hence, G_3^* has an atom at $s = 0$. ∎

Given payoffs u_1^* and u_2^* in an equilibrium, let $\bar{G}_1$ be a mixed strategy such that

$$v_1 \bar{G}_1(s) + v_2(1 - \bar{G}_1(s)) - c_2 s \quad = \quad u_2^* \tag{15}$$

and $\bar{G}_2(s)$ a mixed strategy such that

$$v_1 \bar{G}_2(s) + v_2(1 - \bar{G}_2(s)) - c_1 s \quad = \quad u_1^* \tag{16}$$

Suppose player 3 is absent, $\bar{G}_i$ for $i = 1, 2$ is the strategy of player i such that the other player j's payoff is u_j^* by choosing s. In an equilibrium, $(\underline{s}_1^*, \bar{s}_3^*)$ may be empty.[24] If the interval is not empty, the following result compares $\bar{G}_i$ with equilibrium strategy G_i^*, and shows that $\bar{G}_i$ is "more aggressive" than G_i^* in terms of first order stochastic dominance.

Lemma 7 *In every equilibrium, for any $s \in (\underline{s}_1^*, \bar{s}_3^*)$, $\bar{G}_1(s) < G_1^*(s)$ and $\bar{G}_2(s) < G_2^*(s)$.*

Proof. According to Lemma 6, any player $i \geq 4$ chooses zero with probability 1. As a result, Lemma 1, the property of "No Aggregate Gaps", implies that at least two of players 1, 2, 3's strategies' supports contain $s \in (\underline{s}_1^*, \bar{s}_3^*)$, and Lemma 3, the property of "Nested Gaps", implies

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[24]See the Case I equilibrium in Proposition 3.

that 1 and 2 must be among these players. Therefore,

$$U_1(s|G_2^*, G_3^*) = u_1^* \tag{17}$$

$$U_2(s|G_1^*, G_3^*) = u_2^* \tag{18}$$

By the definition of $\bar{s}_3^*$, we have $G_3^*(s) < 1$ for $s < \bar{s}_3^*$. Notice that (17) and (18) implicitly define $G_1^*(s)$ and $G_2^*(s)$ as strictly increasing functions of $G_3^*(s)$. Then, if we replace $G_3^*(s)$ with a higher value 1, $G_1^*(s)$ and $G_2^*(s)$ implicitly defined in (17) and (18) should be lower. We can verify that (17) and (18) become (16) and (15) if we replace $G_3^*(s)$ with 1. Therefore, $\bar{G}_1(s) < G_1^*(s)$ and $\bar{G}_2(s) < G_2^*(s)$. ∎

Proof of Proposition 1. We claim that in every equilibrium, given others' equilibrium strategies, player 3's expected value of prizes at $\bar{s}_3^*$ is x_3 as in Definition i). We prove the claim in two steps.

First, in every equilibrium, $\bar{s}_3^* = \inf \arg\max_{s \in [\underline{s}_1, \bar{s}_1]} U_3(s|\bar{G}_1, \bar{G}_2)$, where $\bar{G}_1(s)$ and $\bar{G}_2(s)$ are defined in Lemma 7 and $\underline{s}_1, \bar{s}_1$ solve $\bar{G}_1(\underline{s}_1) = 0$ and $\bar{G}_1(\bar{s}_1) = 1$. To see why, notice that $\bar{G}_i(s) = G_i^*(s)$ for any $s \geq \bar{s}_3^*$ and $i = 1, 2$, so the definition of equilibrium implies

$$u_3^* = U_3(\bar{s}_3^*|\bar{G}_1, \bar{G}_2) \geq U_3(s|\bar{G}_1, \bar{G}_2) \tag{19}$$

for any $s \geq \bar{s}_3^*$. Therefore, $\inf \arg\max_{s \in [\underline{s}_1, \bar{s}_1]} U_3(s|\bar{G}_1, \bar{G}_2) \leq \bar{s}_3^*$. As a result, it remains to be shown that any $s \in [\underline{s}_1, \bar{s}_3^*)$ does not maximize $U_3(s|\bar{G}_1, \bar{G}_2)$.

Lemma 7 implies that if $\bar{G}_1(s) = 0$, then $G_1^*(s) > 0$. Therefore, $\underline{s}_1^* < \underline{s}_1$. The lemma also implies $U_3(s|\bar{G}_1, \bar{G}_2) < U_3(s|G_1^*, G_2^*) \leq u_3^*$ for $s \in (\underline{s}_1^*, \bar{s}_3^*)$. Because $[\underline{s}_1, \bar{s}_3^*)$ is a subset of $(\underline{s}_1^*, \bar{s}_3^*)$, we have $U_3(s|\bar{G}_1, \bar{G}_2) < u_3^*$ for $s \in [\underline{s}_1, \bar{s}_3^*)$. Recall that $u_3^* = U_3(\bar{s}_3^*|\bar{G}_1, \bar{G}_2)$ in (19), so $U_3(s|\bar{G}_1, \bar{G}_2) < U_3(\bar{s}_3^*|\bar{G}_1, \bar{G}_2)$ for $s \in [\underline{s}_1, \bar{s}_3^*)$. Hence, any $s \in [\underline{s}_1, \bar{s}_3^*)$ does not maximize $U_3(s|\bar{G}_1, \bar{G}_2)$.

Second, given others' equilibrium strategies, player 3's expected prize at $\bar{s}_3^*$ equals x_3 as in Definition i). To see why, notice that Lemma 4, the "Stochastic Dominance" property, implies $\bar{s}_i^* \geq \bar{s}_3^*$ for $i = 1, 2$. In addition, we must have $\bar{s}_1^* = \bar{s}_2^*$ because of Lemma 1, the property of "No Aggregate Gaps". Therefore, $u_i^* = v_1 - c_i \bar{s}_1^*$ for $i = 1, 2$. Substituting them into (16) and (15), we can verify that

$$G_i^2(s) = \bar{G}_i(s + \bar{s}_1^* - \bar{s}_1^2) \tag{20}$$

for $i = 1, 2$. That is, $\bar{G}_i$ is G_i^2 shifted horizontally by $\bar{s}_1^* - \bar{s}_1^2$ for $i = 1, 2$. Therefore, the maximizer $\bar{s}_3^* = \inf \arg\max_{s \in [\underline{s}_1, \bar{s}_1]} U_3(s|\bar{G}_1, \bar{G}_2)$ is the maximizer $\hat{s}_3 = \inf \arg\max_{s \in [0, \bar{s}_1^2]} U_3(s|G_1^2, G_2^2)$ with the same shift. That is,

$$\hat{s}_3 = \bar{s}_3^* - (\bar{s}_1^* - \hat{s}_1) \tag{21}$$

Substituting (20) and (21) into the definition $x_3 = U_3(\hat{s}_3|G_1^2, G_2^2) + c_3 \hat{s}_3$, we have $x_3 = U_3(\bar{s}_3^*|\bar{G}_1, \bar{G}_2) + c_3 \bar{s}_3^*$. Recall that $\bar{G}_i(s) = G_i^*(s)$ for $s \geq \bar{s}_3^*$, so $x_3 = U_3(\bar{s}_3^*|G_1^*, G_2^*) + c_3 \bar{s}_3^*$, which is exactly player 3's expected value of prizes at $\bar{s}_3^*$ given others' equilibrium strategies. Hence, we prove the

claim.

Therefore, $T = \bar{s}_3^*$ and $w_3 = 0$. Recall that Lemma 6 implies $u_3^* = 0$, so $u_3^* = \max(0, w_3)$. It remains to be shown that $u_i^* = \max(0, w_i)$ for $i = 1, 2$. As in the second step above, we can show that $x_i = v_1 G_j^*(\bar{s}_3^*) + v_2(1 - G_j^*(\bar{s}_3^*)) - c_i \bar{s}_3^*$ for $i, j \in \{1, 2\}$ and $i \neq j$. The definition of w_i implies $w_i = x_i - c_i T = v_1 G_j^*(\bar{s}_3^*) + v_2(1 - G_j^*(\bar{s}_3^*)) - c_i \bar{s}_3^* = u_i^*$ for distinct $i, j \in \{1, 2\}$. ∎

Having proved Proposition 1, we can use it and Definitions i) to iii) to derive the closed-form expressions of equilibrium payoffs in Proposition 2.

Proof of Proposition 2. We have shown $u_3^* = 0$, so it remains to derive the payoffs for 1 and 2. Consider Case II first. Recall that $U_3(\cdot|G_1^2, G_2^2)$ is maximized at the upper boundary $\bar{s}_1^2$ if (9) does not hold but (10) does. Therefore, Definitions i) to iii) imply $x_i = v_1$ for $i = 1, 2, 3$ and $r_3 = v_1/c_3$, so $u_i^* = v_1(1 - c_i/c_3)$ for $i = 1, 2$.

Consider Case I. Recall that the lower boundary 0 is a maximizer of $U_3(\cdot|G_1^2, G_2^2)$ over $[0, \bar{s}_1^2]$ if neither (9) nor (10) holds. Therefore, Definitions i) to iii) imply $x_1 = v_1 - (v_1 - v_2)c_1/c_2$, $x_2 = v_2$ and $x_3 = v_2(1 - c_1/c_2)$. In addition, $x_3 - c_3 T = 0$, so $T = v_2(1 - c_1/c_2)/c_3$. The definitions also imply $u_i^* = x_i - c_i T$ for $i = 1, 2$. Substituting x_1, x_2 and T into this expression, we obtain the expressions of u_1^* and u_2^* in Case I.

Consider Case III. Recall that (9) implies $U_3(\cdot|G_1^2, G_2^2)$ has an interior maximizer in $[0, \bar{s}_1^2]$. Rearranging terms, we have

$$U_3(s|G_1^2, G_2^2) = \frac{\Delta_1 - \Delta_2}{\Delta_1^2} c_1 c_2 s^2 + \left[\frac{\Delta_1 - \Delta_2}{\Delta_1}(c_2 - c_1) + \frac{\Delta_2}{\Delta_1}(c_1 + c_2) - c_3 \right] s + \Delta_2 \left(1 - \frac{c_1}{c_2} \right)$$

which is a quadratic function of s with its maximum being

$$\hat{u}_3 = \Delta_2 \left(1 - \frac{c_1}{c_2} \right) - \frac{c_1 c_2}{4}(\Delta_1 - \Delta_2) \left[\frac{c_2 - c_1}{c_1 c_2} + \frac{c_1 + c_2}{c_1 c_2} \frac{\Delta_2}{\Delta_1 - \Delta_2} - \frac{\Delta_1}{\Delta_1 - \Delta_2} \frac{c_3}{c_1 c_2} \right]^2$$

Substituting $\Delta_1 = v_1 - v_2$ and $\Delta_2 = v_2$ into the above expression, we obtain (11). Proposition 1 implies $x_3 - c_3 T = u_3^* = 0$, and the definition of x_3 implies $x_3 - c_3 \hat{s}_3 = \hat{u}_3$. Therefore,

$$T - \hat{s}_3 = \hat{u}_3/c_3 \tag{22}$$

Then, for $i = 1, 2$, we have $u_i^* = x_i - c_i T = u_i^2 + c_i \hat{s}_3 - c_i T = u_i^2 - \hat{u}_3 c_i/c_3$, where the first equality is from Proposition 1, the second from the definition of x_i and the last from (22). Substituting $u_i^2 = v_1 - \Delta_1 c_i/c_2$ into the above expression of u_i^*, we obtain the payoff expressions in Case III. ∎

Proof of Corollary 1. We can verify that the equilibrium payoffs in Proposition 2 are continuous in Δ_1/Δ_2, so it is sufficient to prove the corollary in each of Case I, II, and III. Consider Case II first. Using Proposition 2, we can verify that $(u_1^* - u_2^*)/(u_2^* - u_3^*)$ is independent of $\mathbf{v}$, so it is nondecreasing in Δ_1/Δ_2.

Consider Case I. Substituting the payoff expressions in Proposition 2 into $(u_1^* - u_2^*)/(u_2^* - u_3^*)$, we can rewrite it as

$$\frac{u_1^* - u_2^*}{u_2^* - u_3^*} = \left[\frac{\Delta_1}{\Delta_2} \left(1 - \frac{c_1}{c_2} \right) + 1 - \frac{c_1}{c_3} \left(1 - \frac{c_1}{c_2} \right) \right] \bigg/ \left[1 - \frac{c_2}{c_3} \left(1 - \frac{c_1}{c_2} \right) \right]$$

which is strictly increasing in Δ_1/Δ_2.

Consider Case III. In the proof of Proposition 1, we show that in any equilibrium, $u_i^* = v_1 - c_i \bar{s}_1^*$ for $i = 1, 2$. Then, using the expressions of u_1^*, u_2^* and $u_3^* = 0$, we can rewrite

$$\frac{u_1^* - u_2^*}{u_2^* - u_3^*} = \frac{v_1/v_2 - c_1 \bar{s}_1^*/v_2}{v_1/v_2 - c_2 \bar{s}_1^*/v_2} - 1$$

Notice that $(u_1^* - u_2^*)/(u_2^* - u_3^*)$ being nondecreasing in Δ_1/Δ_2 is equivalent to $v_1/v_2 - c_1 \bar{s}_1^*/v_2 - (v_1/v_2 - c_2 \bar{s}_1^*/v_2) = (c_2 - c_1) \bar{s}_1^*/v_2$ being nondecreasing in Δ_1/Δ_2. Hence, it is sufficient to show that $\bar{s}_1^*/v_2$ is nondecreasing in Δ_1/Δ_2.

Recall that $u_1^* = v_1 - c_1 \bar{s}_1^*$, so $\bar{s}_1^*/v_2 = (v_1 - u_1^*)/(c_3 v_2)$. In addition, because of the expression of u_1^* in Proposition 2 and (11), we can rewrite $\bar{s}_1^*/v_2$ as

$$\frac{\bar{s}_1^*}{v_2} = \frac{1}{c_3} \left(1 - \frac{c_1}{c_2} \right) + \frac{\Delta_1}{\Delta_2} \frac{1}{c_2} + \frac{1}{4 c_1 c_2 c_3} \frac{[(c_3 - c_2 - c_1) - (1 - \Delta_1/\Delta_2)(c_3 + c_1 - c_2)]^2}{1 - \Delta_1/\Delta_2}$$

Consider the last term. Its denominator is decreasing in Δ_1/Δ_2. Its numerator is increasing in Δ_1/Δ_2 because $c_3 - c_2 - c_1 > 0$ and $c_3 + c_1 - c_2 > 0$ in Case III. Therefore, the last term is increasing in Δ_1/Δ_2, and hence, so is $\bar{s}_1^*/v_2$. ∎

Next, we present an algorithm, which is used to prove Propositions 3 to 6. Using the equilibrium payoffs derived in Proposition 2, the algorithm below constructs a strategy profile $\mathbf{G} = (G_1, G_2, G_3)$.

Algorithm:

Step 1. Define $\mathbf{G}$ at the lowest performance $s = 0$: $G_1(0) = G_2(0) = 0, G_3(0) = u_2^*/v_2$.

Step 2. This step examines $\mathbf{G}(0)$ to determine $\mathcal{A}^+(0)$, the set of players whose strategies are increasing at performance $s = 0$. This step contains two parts:

    Part One. Define a set of candidates $\mathcal{CP}(s) = \{i \in \{1, 2, 3\}$ such that $U_i(s|G_j, G_k) = u_i^*$ for distinct $i, j, k \in \{1, 2, 3\}\}$.

    Part Two. This part refines the candidate set to $\mathcal{A}^+(s)$: Consider an equation system

$$\begin{bmatrix} 0 & K_3(s) & K_2(s) \\ K_3(s) & 0 & K_1(s) \\ K_2(s) & K_1(s) & 0 \end{bmatrix} \begin{bmatrix} g_1(s) \\ g_2(s) \\ g_3(s) \end{bmatrix} = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} \tag{23}$$

    where $K_i(s) = (\Delta_1 - \Delta_2)G_i(s) + \Delta_2$ for $i = 1, 2, 3$. If the solution $g_3(s) > 0$, let $\mathcal{A}^+(s) = \mathcal{CP}(s)$; otherwise $\mathcal{A}^+(s) = \mathcal{CP}(s) \backslash \{3\}$.

Step 3. Given $\mathbf{G}(0)$ and $\mathcal{A}^+(0)$ defined above, this step extends the definition of $\mathbf{G}$ to performance higher than $s = 0$. Specifically, for $i \in \mathcal{A}^+(s)$ and t slightly higher than s, consider the equation system $U_i(t|G_j, G_k) = u_i^*$ for $j, k \in \mathcal{A}^+(s)\backslash\{i\}$. We can solve $G_i(t) \in [0, 1]$ for $i \in \mathcal{A}^+(s)$ from the system.[25] If $i \notin \mathcal{A}^+(s)$, let $G_i(t) = G_i(s)$. In this way, we can extend $\mathbf{G}$ to performance above s until we reach a switch point s'.

Step 4. This step determines the switch point and extends the definition of $\mathbf{G}$ above the switch point. The switch point s' is the lowest performance above s for which $\mathcal{A}^+(s) \neq \mathcal{A}^+(s')$. This happens for two reasons. The first reason is a player outside of $\mathcal{A}^+(s)$ enters the set at s' according to Step 2. The second reason is a player in $\mathcal{A}^+(s)$ exits the set at s' according to Step 2. Then, given $\mathbf{G}(s')$ and $\mathcal{A}^+(s')$, continue extending $\mathbf{G}$ to higher performance as in Step 3 until another switch point is reached. In this way, we can extend the definition of $\mathbf{G}$ to higher performance until $s = (v_1 - u_1^*)/c_1$ is reached.[26]

The above algorithm is simpler than that of Siegel (2010), which is due to the constant marginal costs. The main difference from his algorithm is Part Two of Step 2. In contrast to Siegel's algorithm, Part Two in our algorithm does not use a fixed point argument. Specifically, suppose $\mathcal{CP}(s) = \{1, 2, 3\}$ from Part One. Then, the definition of $\mathcal{CP}(s)$ implies $U_i(s|G_j, G_k) = u_i^*$ for $j, k \in \{1, 2, 3\}\backslash\{i\}$. If all candidates' strategies in an equilibrium are indeed increasing at s, the above equation remains true for bids slightly above s. Differentiating both sides of the equation with respect to s, we obtain $K_j(s)g_k(s) + K_k(s)g_j(s) = c_i$, where $g_i(s)$ is the derivative of G_i.[27] The matrix form of these equations is (23) in Part Two. Therefore, if all candidates' strategies are indeed increasing at s in an equilibrium, (23) must have positive solutions $g_i(s) > 0$ for $i = 1, 2, 3$. If $g_3(s) \leq 0$, then at least one candidate's strategy is not increasing at s. It turns out this candidate must be player 3, who has the highest marginal cost. So far we explain Part Two in the case of $\mathcal{CP}(s) = \{1, 2, 3\}$ and $g_3(s) \leq 0$, and this case turns out to be the only case in which $\mathcal{CP}(s)$ is different from $\mathcal{A}^+(s)$.[28]

Proof of Proposition 3. We first use the algorithm to derive a set of strategies, then show that it is the unique equilibrium. Following the algorithm, $\mathcal{A}^+(0) = \{2, 3\}$, then we can extend G_2, G_3 by solving $v_2 G_i(s) - c_j s = u_i^*$ for $i, j \in \{2, 3\}$ and $j \neq i$. Notice that (9) and (10) imply that u_2^*, u_3^* are as in Case I in Proposition 2. Substituting the payoffs into the two equations above, we have

$$
\begin{aligned}
G_2(s) &= sc_3/v_2 \\
G_3(s) &= sc_2/v_2 + (c_3 - c_2 + c_1)/c_3
\end{aligned}
$$

for $s \in [0, s']$ where $s' = (c_2 - c_1)v_2/(c_2 c_3)$ solves $G_3(s') = 1$. We can verify that the first switch

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[25]In the proofs of Propositions 3-6, we solve the equation system in four different cases, and show that it has a unique solution.

[26]According to the proofs of Propositions 3-6, there are at most two switch points above 0.

[27]The derivatives exist because of the implicit function theorem.

[28]This is because of the "Nested Gaps" property in Lemma 3.

point above 0 is s', and $\mathcal{A}^+(s') = \{1, 2\}$, so we can extend G_1, G_2 by solving

$$v_1 G_2(s) + v_2(1 - G_2(s)) - c_1 s = u_1^* \tag{24}$$

$$v_1 G_1(s) + v_2(1 - G_1(s)) - c_2 s = u_2^* \tag{25}$$

Substituting expressions of u_1^* and u_2^* in Case I, we can solve the above equations and get

$$
\begin{aligned}
G_2(s) &= \frac{c_1}{\Delta_1} s + 1 - \frac{c_1}{c_2} - \frac{c_1}{c_3}\left(1 - \frac{c_1}{c_2}\right)\frac{\Delta_2}{\Delta_1} \\
G_1(s) &= \frac{c_2}{\Delta_1} s - \frac{\Delta_2}{\Delta_1}\frac{c_2 - c_1}{c_3}
\end{aligned}
$$

for $s \in [s', s'']$ where $s'' = \Delta_1/c_2 + \Delta_2(c_2 - c_1)/(c_2 c_3)$ solves $G_1(s'') = 1$.

It is straightforward to verify that (G_1, G_2, G_3) is indeed an equilibrium, and we show below that there are no other equilibria. First, in any equilibrium, G_i^* for $i = 2, 3$ must satisfy $v_2 G_i(s) - c_j s = u_j^*$ for $j \in \{2, 3\} \setminus \{i\}$ and $s \in [0, \underline{s}_1^*]$, where $\underline{s}_1^*$ is the lower boundary of G_i^*'s support. Therefore, $G_i^*(s) = G_i(s)$ for $i = 2, 3$ and for $s \in [0, \underline{s}_1^*]$. At $\underline{s}_1^*$, we have $U_1(\underline{s}_1^* | G_2^*, G_3^*) = u_1^*$. The definition of switch point s' implies that it is the lowest bid satisfying this property, so $\underline{s}_1^* = s'$. Notice that $G_3^*(s') = 1$, so $\underline{s}_1^* = \bar{s}_3^* = s'$. Hence, G_i^* for $i = 1, 2$ satisfies (24) and (25) for $s > s'$. Therefore, $G_i^*(s) = G_i(s)$ for $i = 1, 2$ and for $s \in [s', s'']$. As above, from the equilibrium payoffs, we uniquely determine the strategies in any equilibrium. Therefore, there are no other equilibria. ■

Proof of Proposition 4. As in Proposition 3, we first use the algorithm to construct a set of strategies. With $\mathcal{A}^+(0) = \{2, 3\}$, we can extend G_2 and G_3 by solving $v_2 G_i(s) - c_j s = u_j^*$ for $i, j \in \{2, 3\}$ and $j \neq i$. That is, $G_2(s) = sc_3/v_2$ and $G_3(s) = sc_2/v_2 + (1 - c_2/c_3) v_1/v_2$.

Denote the next switch point as s'. By its definition, s' is the smallest bid such that $U_1(s | G_2, G_3) = u_1^*$. Therefore, we can extend G_1, G_2, G_3 by solving $U_i(s | G_j, G_k) = u_i^*$ for $i = 1, 2, 3$. If $\Delta_1/\Delta_2 = 1$, we can solve the linear equation system and obtain $G_i(s) = \sum_{j \neq i}(u_j^* + c_j s)/(2v_2)$ for $i = 1, 2, 3$. If $\Delta_1/\Delta_2 \neq 1$, we can use the definition of A_i to rewrite $U_i(s | G_j, G_k) = u_i^*$ as

$$\left(G_j(s) + \frac{\Delta_2}{\Delta_1 - \Delta_2}\right)\left(G_k(s) + \frac{\Delta_2}{\Delta_1 - \Delta_2}\right) = \frac{A_i + c_i s}{\Delta_1 - \Delta_2} \tag{26}$$

The product of (26) for $i = 1, 2, 3$ is

$$\left[\times_{i=1}^{3}\left(G_i(s) + \frac{\Delta_2}{\Delta_1 - \Delta_2}\right)\right]^2 = \times_{i=1}^{3}\left(\frac{A_i + c_i s}{\Delta_1 - \Delta_2}\right)$$

therefore

$$\times_{i=1}^{3}\left(G_i(s) + \frac{\Delta_2}{\Delta_1 - \Delta_2}\right) = \pm\sqrt{\times_{i=1}^{3}\left(\frac{A_i + c_i s}{\Delta_1 - \Delta_2}\right)} \tag{27}$$

Combining (26) and (27), we obtain

$$G_i(s) = \pm \frac{\Delta_1 - \Delta_2}{A_i + c_i s} \sqrt{\times_{i=1}^{3}\left(\frac{A_i + c_i s}{\Delta_1 - \Delta_2}\right)} - \frac{\Delta_2}{\Delta_1 - \Delta_2}$$

On the one hand, if $\Delta_1 - \Delta_2 > 0$, we can verify that $A_1 + c_1 s > A_2 + c_2 s > A_3 + c_3 s > 0$ for $s \in (0, v_1/c_3)$, so

$$G_i(s) = \frac{1}{A_i + c_i s} \sqrt{\frac{\times_{i=1}^{3}(A_i + c_i s)}{\Delta_1 - \Delta_2}} - \frac{\Delta_2}{\Delta_1 - \Delta_2} \tag{28}$$

otherwise $G_i(s) < 0$. Notice that $A_i + c_i s$ and $\Delta_1 - \Delta_2$ are both positive, so the square root in (28) is a real number. Therefore, $F_i(s)$, which is the right hand side of (28), is also a real number. On the other hand, if $\Delta_1 - \Delta_2 < 0$, we can verify that $A_3 + c_3 s < A_2 + c_2 s < A_1 + c_1 s < 0$ for $s \in (0, v_1/c_3)$. Moreover, the second term in (28) satisfies $-\Delta_2/(\Delta_1 - \Delta_2) > 1$. Then, (28) is also true, otherwise $G_i(s) > 1$. Notice that $A_i + c_i s$ and $\Delta_1 - \Delta_2$ are negative, so the square root in (28) is a real number, so is $F_i(s)$. Hence, the strategies in Proposition 4 are the unique outcome of the algorithm.

Let us verify that G_i is nondecreasing for $i = 1, 2, 3$. By the construction in the algorithm, G_i is also continuous. It is straightforward to verify that the linear parts of G_i are nondecreasing. Therefore, it remains to verify that G_i is nondecreasing over $s \in (\underline{s}_1^*, v_1/c_3)$. First, consider the case with $\Delta_1 - \Delta_2 > 0$. Recall that in this case, $A_1 + c_1 s > A_2 + c_2 s > A_3 + c_3 s > 0$ for $s \in (0, v_1/c_3)$, so

$$G_3(s) = \sqrt{\frac{(A_1 + c_1 s)(A_2 + c_2 s)}{(A_3 + c_3 s)(\Delta_1 - \Delta_2)}} - \frac{\Delta_2}{\Delta_1 - \Delta_2}$$

When $\Delta_1 - \Delta_2 > 0$, $G_3(s)$ is a monotone transformation of $(A_1 + c_1 s)(A_2 + c_2 s)/(A_3 + c_3 s)$. Therefore, (14) implies the right derivative $G_3'(\underline{s}_1^*) \geq 0$.[29] In addition,

$$\frac{\partial}{\partial s}\left[\frac{(A_1 + c_1 s)(A_2 + c_2 s)}{A_3 + c_3 s}\right] = \frac{c_1 c_2 c_3 (s^2 + 2\frac{A_3}{c_3}s) + c_1 A_2 A_3 + A_1 c_2 A_3 - A_1 A_2 c_3}{(A_3 + c_3 s)^2} \tag{29}$$

where the denominator is positive and the numerator is increasing in s. The positive denominator implies that $G_3'(s)$ and the numerator have the same sign. Recall that the numerator is increasing in s and it is nonnegative at $\underline{s}_1^*$ due to (14), so it is positive for $s > \underline{s}_1^*$. Therefore, $G_3'(s) > 0$ for $s > \underline{s}_1^*$. Moreover, (23) can be rewritten as

$$((v_1 - 2v_2)G_3(s) + v_2)g_2(s) + ((v_1 - 2v_2)G_2(s) + v_2)g_3(s) = c_1 \tag{30}$$

$$((v_1 - 2v_2)G_3(s) + v_2)g_1(s) + ((v_1 - 2v_2)G_1(s) + v_2)g_3(s) = c_2 \tag{31}$$

$$((v_1 - 2v_2)G_2(s) + v_2)g_1(s) + ((v_1 - 2v_2)G_1(s) + v_2)g_2(s) = c_3 \tag{32}$$

Recall that $A_1 + c_1 s > A_2 + c_2 s > A_3 + c_3 s > 0$ for $s \in (0, v_1/c_3)$ if $\Delta_1 - \Delta_2 > 0$. Therefore, (28) implies $G_1(s) < G_2(s) < G_3(s)$ for $s \in (0, v_1/c_3)$. Comparing (30) and (31), we obtain

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[29]We use the right derivative because, for $s < \underline{s}_1^*$, the expression of $G_3(s)$ is different.

$g_2(s) < g_1(s)$. Similar, comparing (31) and (32), we obtain $g_3(s) < g_2(s)$. Therefore, $0 < G_3'(s) < G_2'(s) < G_1'(s)$.

We have verified that G_i is nondecreasing if $\Delta_1 - \Delta_2 > 0$. Next, we consider the case with $\Delta_1 - \Delta_2 < 0$. As above, we have $G_3'(s) < G_2'(s) < G_1'(s)$, so it remains to show $G_3'(s) > 0$ for $s \in (\underline{s}_1^*, v_1/c_3)$. Recall that, if $\Delta_1 - \Delta_2 < 0$, we have $A_3 + c_3 s < A_2 + c_2 s < A_1 + c_1 s < 0$ for $s \in (0, v_1/c_3)$, so

$$G_3(s) = -\sqrt{\frac{(A_1 + c_1 s)(A_2 + c_2 s)}{(A_3 + c_3 s)(\Delta_1 - \Delta_2)}} - \frac{\Delta_2}{\Delta_1 - \Delta_2}$$

Using the same argument in the case of $\Delta_1 - \Delta_2 > 0$, we obtain that $G_3'(s)$ and the numerator in (29) also have the same sign for $s \in (\underline{s}_1^*, v_1/c_3)$. Notice that the numerator is a quadratic function that reaches its minimum at $s = -A_3/c_3 = (\Delta_2)^2/((\Delta_2 - \Delta_1)c_3) > v_1/c_3$, so it is decreasing in s for $s \in (\underline{s}_1^*, v_1/c_3)$. Notice that $G_i(v_1/c_3) = 1$, so (30)-(32) imply the left derivative $G_3'(v_1/c_3) = (c_1 + c_2 - c_3)/(2\Delta_1) \geq 0$, where the inequality is from the assumption $c_1 + c_2 \geq c_3$ of the proposition. As a result, the numerator in (29) is nonnegative at $s = v_1/c_3$, so it is positive over for $s \in (\underline{s}_1^*, v_1/c_3)$ because it is decreasing in s. Hence, (29) is positive and G_3 is increasing.

Now we verify that the constructed strategies are a unique equilibrium. As in Proposition 3, $G_i^*(s) = G_i(s)$ for $s \leq s'$. We show $G_i(s) = G_i^*(s)$ for $i = 1, 2, 3$ and for $s > s'$ in two steps.

First, $\underline{s}_1^* = s'$. As in Proposition 3, the definition of s' implies that $\underline{s}_1^* \geq s'$. Suppose $\underline{s}_1^* > s'$, then $G_2^*(s), G_3^*(s)$ satisfy $v_2 G_2^*(s) - c_3 s = u_3^*$ and $v_2 G_3^*(s) - c_2 s = u_2^*$ for $s \in (s', s'')$. Following the same argument in the proof of Lemma 7, we can verify that $G_2^*(s) > G_2(s)$ and $G_3^*(s) > G_3(s)$ for $s \in (s', s'')$. Therefore, $U_1(s|G_2^*, G_3^*) > U_1(s|G_2, G_3) = u_1^*$ for $s \in (s', s'')$, where the equality is from the definition of G_2, G_3. Hence, $\underline{s}_1^* = s'$.

Second, any $s \in [s', s'']$ is in the support of G_i^* for $i = 1, 2, 3$. Suppose otherwise that there exists $\varepsilon > 0$ and $s_3 \in [s', s'']$ such that $(s_3, s_3 + \varepsilon)$ is not a subset of G_i^*'s support. Then, Lemma 3, the property of "Nested Gaps", implies that G_i^* must be G_3^*. Without loss of generality, assume s_3 is the smallest bid with the above property. Then, $G_3^*(s) = G_3^*(s_3)$ for $s \in [s_3, s_3 + \varepsilon]$. Moreover, $G_3(s) > G_3^*(s)$ because G_3 is increasing and $G_3(s_3) = G_3^*(s)$. Then, as in the proof of Lemma 7, we have $G_i(s) < G_i^*(s)$ for $i = 1, 2$ and $s \in [s_3, s_3 + \varepsilon]$. Therefore, $U_3(s|G_1^*, G_2^*) > U_3(s|G_1, G_2) = u_3^*$ for $s \in [s_3, s_3 + \varepsilon]$. This is a contradiction.

In these two steps, we verify that G_i^* and G_i have the same support for $i = 1, 2, 3$. Moreover, the construction above shows that, given the equilibrium payoffs, G_i is the unique strategy with such support. Hence, there are no other equilibria. ∎

Proof of Proposition 5. The algorithm implies that there are two switch points between 0 and $(v_1 - u_1^*)/c_1$, where the algorithm ends. Moreover, they satisfy $s' < s''$, and $\mathcal{A}^+(s') = \{1, 2, 3\}$ and $\mathcal{A}^+(s'') = \{1, 2\}$. Following the calculations in the proof of Proposition 4, we can construct the strategies in this proposition.

Let us verify that the constructed G_i are indeed non-decreasing. We first verify that the left derivative $G_3'(\bar{s}_3^*) = 0$. Notice that this proposition corresponds to Case III in Proposition

2, so $U_3(\cdot|G_1^2, G_2^2)$ has an interior maximizer over $[0, \bar{s}_1^2]$. In addition, according to the proof of Proposition 1, $U_3(\cdot|G_1^2, G_2^2)$ is $U_3(\cdot|\bar{G}_1, \bar{G}_2)$ with a horizontal shift. Therefore, $U_3(\cdot|\bar{G}_1, \bar{G}_2)$ also has an interior maximizer over $[\underline{s}_1, \bar{s}_1]$, and the maximizer is $\bar{s}_3^*$. The corresponding first order condition is (32) with $s = \bar{s}_3^*$, which, according to the proof of Proposition 1, equals to the threshold T. We can also verify that $g_1(\bar{s}_3^*) = c_2/\Delta_1$, $g_2(\bar{s}_3^*) = c_1/\Delta_1$ and $g_3(\bar{s}_3^*) = 0$ solve (30)-(32), so the left derivative $G_3'(\bar{s}_3^*) = g_3(\bar{s}_3^*) = 0$.

Second, G_3 is nondecreasing over $(\underline{s}_1^*, \bar{s}_3^*)$. Recall that for $s \in (\underline{s}_1^*, \bar{s}_3^*)$, $G_3'(s)$ and the numerator in (29) have the same sign. Notice that $G_3(\bar{s}_3^*) = 1$, so $G_3(s) < 1$ implies $G_3'(s) \geq 0$ for s slightly below $\bar{s}_3^*$. Therefore, the numerator is also nonnegative at s slightly below $\bar{s}_3^*$. In addition, the numerator is zero at $s = \bar{s}_3^*$ due to the first step. Therefore, the numerator, which is a U-shaped quadratic function of s, is positive for $s < \bar{s}_3^*$. Hence, $G_3'(s) > 0$ for $s \in (\underline{s}_1^*, \bar{s}_3^*)$.

As in the proof of Proposition 4, $G_1'(s) > G_2'(s) > G_3'(s)$ for $s \in (\underline{s}_1^*, \bar{s}_3^*)$, so G_i is increasing over $(\underline{s}_1^*, \bar{s}_3^*)$. The other parts of the strategies are linear, and are also non-decreasing.

By the two steps in the proof of Proposition 4, we can prove the constructed strategies are a unique equilibrium. ∎

Proof of Proposition 6. The algorithm implies that there are two switch points between 0 and $(v_1 - u_1^*)/c_1$. Moreover, $s' < s''$, and $\mathcal{A}^+(s') = \{1, 2\}$ and $\mathcal{A}^+(s'') = \{1, 2, 3\}$. As in the proof of Proposition 4, we have $G_i^*(s) = G_i(s)$ for $s \leq s'$. At the first switch point s', we have $g_3(s') < 0$, so at least one of G_1^*, G_2^*, G_3^*'s supports does not contain $(s', s' + \varepsilon)$ for some $\varepsilon > 0$. Then, the property of "Nested Gaps" in Lemma 3 implies that $(s', s' + \varepsilon)$ is not a subset of G_3^*'s support. By the definition of the second switch point s'', (s', s'') cannot be in the support of G_3^*. By the same argument for $[s', s'']$ in Proposition 4, we have $[s'', (v_1 - u_1^*)/c_1]$ is in the support of G_i^* for $i = 1, 2, 3$.

Let us verify that G_i is nondecreasing. As above, it is sufficient to verify that G_3 is nondecreasing for its nonlinear part over $(\hat{s}_3, \bar{s}_1^*)$. As in the proof of Proposition 4, $\Delta_1 - \Delta_2 > 0$ implies that the numerator in (29) is increasing over $(0, \bar{s}_1^*)$. Because $F_3'(s)$ has the same sign as the numerator, so $F_3(s)$ is a U-shaped function over $(0, \bar{s}_1^*)$. By their definitions, $F_i(\underline{s}_1^*)$ for $i = 1, 2, 3$ are the unique solution in $[0, 1]^3$ to the system $U_1(\underline{s}_1^*|F_2, F_3) = u_1^*$, $U_2(\underline{s}_1^*|F_1, F_3) = u_2^*$ and $U_3(\underline{s}_1^*|F_1, F_2) = u_3^*$. In addition, $U_1(\underline{s}_1^*|G_2, G_3) = u_1^*$, $U_2(\underline{s}_1^*|G_1, G_3) = u_2^*$ and $U_3(\underline{s}_1^*|G_1, G_2) = u_3^*$. Therefore, $G_i(\underline{s}_1^*) = F_i(\underline{s}_1^*)$. The construction of G_3 implies $G_3(\underline{s}_1^*) = G_3(\hat{s}_3) = F_3(\hat{s}_3)$, so $F_3(\underline{s}_1^*) = F_3(\hat{s}_3)$. Hence, the U-shaped function F_3 is increasing over $(\hat{s}_3, \bar{s}_1^*)$.

The first paragraph shows that we uniquely determine the supports of strategies in any equilibrium. Therefore, the construction implies that given the equilibrium payoffs, there are no other strategies with the same supports. Hence, there are no other equilibria. ∎

We first introduce a lemma below, then use it to prove Proposition 7.

Lemma 8 *If $1 < \Delta_1/\Delta_2 < c_1/(c_3 - c_2)$ and $c_3 < c_1 + c_2$, there exists a unique $\lambda \in (1, c_1/(c_3 - c_2))$ such that $F_3'(\underline{s}_1^*) > 0$ if and only if $\Delta_1/\Delta_2 \in (1, \lambda)$.*

Proof. We prove this in three steps. First, $\underline{s}_1^*/v_1$ increases in κ, where $\kappa \equiv 1/((\Delta_1/\Delta_2)^2 - 1)$. To see why, recall that $\underline{s}_1^*$ solves $G_1^*(\underline{s}_1^*) = 0$. That is,

$$\frac{1}{c_1\underline{s}_1^* + A_1}\sqrt{\frac{\times_{i=1}^3 (c_i\underline{s}_1^* + A_i)}{\Delta_1 - \Delta_2}} - \frac{\Delta_2}{\Delta_1 - \Delta_2} = 0$$

Because $\Delta_1 - \Delta_2 > 0$, we can rewrite it as $(c_2\underline{s}_1^* + A_2)(c_3\underline{s}_1^* + A_3)/(c_1\underline{s}_1^* + A_1) = \Delta_2^2/(\Delta_1 - \Delta_2)$, or

$$c_2c_3\underline{s}_1^{*2} + \left(A_2c_3 + c_2A_3 - \frac{\Delta_2^2}{\Delta_1 - \Delta_2}c_1\right)\underline{s}_1^* + A_2A_3 - \frac{\Delta_2^2}{\Delta_1 - \Delta_2}A_1 = 0 \qquad (33)$$

Recall that $A_i = v_1(1 - c_i/c_3) + \Delta_2^2/(\Delta_1 - \Delta_2)$, so

$$\frac{A_i}{v_1} = 1 - \frac{c_i}{c_3} + \frac{\Delta_2^2}{(\Delta_1 - \Delta_2)(\Delta_1 + \Delta_2)} = 1 - \frac{c_i}{c_3} + \frac{1}{t^2 - 1} \qquad (34)$$

$$\frac{\Delta_2^2}{(\Delta_1 - \Delta_2)v_1} = \frac{1}{t^2 - 1} \qquad (35)$$

where $t \equiv \Delta_1/\Delta_2$. Dividing both sides of (33) by v_1^2 and using (34) and (35), we can rewrite (33) as

$$c_2c_3\left(\frac{\underline{s}_1^*}{v_1}\right)^2 + [c_3 - c_2 + \kappa(c_3 + c_2 - c_1)]\frac{\underline{s}_1^*}{v_1} - \kappa\frac{c_2 - c_1}{c_3} = 0 \qquad (36)$$

where $\kappa = 1/(t^2 - 1)$. Because the last term is negative, the above equation has two roots of different signs. Therefore, the positive root is

$$\frac{\underline{s}_1^*}{v_1} = -\frac{c_3 - c_2 + \kappa(c_3 + c_2 - c_1)}{c_2c_3}$$
$$+ \frac{\sqrt{(c_3 - c_2 + \kappa(c_3 + c_2 - c_1))^2 + 4c_2(c_2 - c_1)\kappa}}{c_2c_3} \qquad (37)$$

Recall that $F_3(s) = \frac{1}{A_3 + c_3s}\sqrt{\frac{\times_{i=1}^3(c_is + A_i)}{\Delta_1 - \Delta_2}} - \frac{\Delta_2}{\Delta_1 - \Delta_2}$. Because $\Delta_1 - \Delta_2 > 0$, $F_3'(\underline{s}_1^*) > 0$ is equivalent to (14). Expanding the derivative in (14), we obtain

$$c_1(A_2 + c_2\underline{s}_1^*)(A_3 + c_3\underline{s}_1^*) + (A_1 + c_1\underline{s}_1^*)c_2(A_3 + c_3\underline{s}_1^*) - (A_1 + c_1\underline{s}_1^*)(A_2 + c_2\underline{s}_1^*)c_3 > 0$$

Dividing both sides by $c_1c_2c_3\underline{s}_1^*$, we obtain

$$\left(\frac{A_2}{\underline{s}_1^*c_2} + 1\right)\left(\frac{A_3}{\underline{s}_1^*c_3} + 1\right) + \left(\frac{A_1}{\underline{s}_1^*c_1} + 1\right)\left(\frac{A_3}{\underline{s}_1^*c_3} + 1\right) - \left(\frac{A_1}{\underline{s}_1^*c_1} + 1\right)\left(\frac{A_2}{\underline{s}_1^*c_2} + 1\right) > 0$$

Notice that $\frac{A_i}{\underline{s}_1^*c_i} = \frac{1 - c_i/c_3 + \kappa}{c_i}\frac{1}{\underline{s}_1^*/v_1}$, so the inequality above is

$$\left(\frac{1 - c_2/c_3 + \kappa}{c_2} + \frac{\underline{s}_1^*}{v_1}\right)\left(\frac{1 - c_3/c_3 + \kappa}{c_3} + \frac{\underline{s}_1^*}{v_1}\right) + \left(\frac{1 - c_1/c_3 + \kappa}{c_1} + \frac{\underline{s}_1^*}{v_1}\right)\left(\frac{1 - c_3/c_3 + \kappa}{c_3} + \frac{\underline{s}_1^*}{v_1}\right)$$
$$- \left(\frac{1 - c_1/c_3 + \kappa}{c_1} + \frac{\underline{s}_1^*}{v_1}\right)\left(\frac{1 - c_2/c_3 + \kappa}{c_2} + \frac{\underline{s}_1^*}{v_1}\right) > 0$$

Collecting terms with respect to $\underline{s}_1^*/v_1$, we get the left-hand side of the above inequality

$$LHS = \left(\frac{\underline{s}_1^*}{v_1}\right)^2 + 4\frac{\kappa}{c_3}\frac{\underline{s}_1^*}{v_1} + \frac{1 - c_1/c_3 + \kappa}{c_1} + \frac{1 - c_2/c_3 + \kappa}{c_2} + \frac{1 - c_3/c_3 + \kappa}{c_3} \qquad (38)$$

Recall that $\underline{s}_1^*/v_1$ solves (36), which can be rewritten as

$$c_2c_3\left(\frac{\underline{s}_1^*}{v_1} + \frac{c_3 - c_2 + \kappa(c_3 + c_2 - c_1)}{2c_1c_2}\right)^2 - \left[\left(\frac{c_3 - c_2 + \kappa(c_3 + c_2 - c_1)}{2c_1c_2}\right)^2 + \kappa\frac{c_2 - c_1}{c_3}\right] = 0$$

If κ increases, $\frac{c_3 - c_2 + \kappa(c_3 + c_2 - c_1)}{2c_1c_2}$ increases and the last term decreases. Then, the quadratic function of $\underline{s}_1^*/v_1$ shifts to the right and downwards. This implies that the larger root $\underline{s}_1^*/v_1$ increases.

Second, LHS in (38) is increasing in κ for $\kappa \in [\underline{\kappa}, +\infty)$ if it is increasing at $\underline{\kappa}$, where

$$\underline{\kappa} = \left.\frac{1}{t^2 - 1}\right|_{t = c_1/(c_3 - c_2)} = \frac{(c_3 - c_2)^2}{(c_1 + c_2 - c_3)(c_1 + c_3 - c_2)}$$

The first step implies that $\underline{s}_1^*/v_1$ increases in κ, so the first two terms in (38) increase in κ. However, the last three terms do not. To see why, the derivative of the last three terms with respect to κ is

$$2\kappa\frac{c_1 + c_2 - c_3}{c_1c_2c_3} - \frac{2(c_3 - c_2)(c_3 - c_1)}{c_1c_2c_3^2}$$

which is increasing in κ. However, the derivative at $\kappa = \underline{\kappa}$ is

$$-\frac{2c_1(c_2 - c_1)(c_3 - c_2)}{c_1c_2c_3^2(c_1 + c_3 - c_2)} < 0 \qquad (39)$$

Therefore, the sum of the last three terms is a U-shaped quadratic function of κ. Hence, to show (38) is increasing in κ, it is sufficient to show it is increasing in κ at $\underline{\kappa}$.

Third, LHS in (38) is increasing in κ at $\underline{\kappa}$. Substituting (39) into the derivative of (38) with respect to κ at $\underline{\kappa}$, we have

$$\left.\frac{\partial LHS}{\partial \kappa}\right|_{\underline{\kappa}} = \left(2y + \frac{4}{c_3}\kappa\right)\left.\frac{\partial y}{\partial \kappa}\right|_{\underline{\kappa}} + \frac{4}{c_3}y - \frac{2c_1(c_2 - c_1)(c_3 - c_2)}{c_1c_2c_3^2(c_1 + c_3 - c_2)} \qquad (40)$$

where $y \equiv \underline{s}_1^*/v_1$. Recall that if κ increases, the quadratic function in (36) shifts to the right and downwards. Therefore, the larger root y of the equation shifts to the right by at least $\frac{c_3 + c_2 - c_1}{2c_2c_3}$. That is, $\left.\frac{\partial y}{\partial \kappa}\right|_{\underline{\kappa}} \geq \frac{c_3 + c_2 - c_1}{2c_2c_3}$. Substituting $\underline{\kappa}$ into (37), we get

$$y(\underline{\kappa}) = \frac{(c_2 - c_1)(c_3 - c_2)}{c_2c_3(c_3 - c_2 + c_1)}$$

Substituting $y(\underline{\kappa})$, $\underline{\kappa}$ and the lower bound of $\partial y(\underline{\kappa})/\partial \kappa$ into (40), then multiplying both sides by

$c_2 c_3^2/2$, we get

$$\left. \frac{\partial LHS}{\partial \kappa} \right|_{\underline{\kappa}} \frac{c_2 c_3^2}{2} = \left(\frac{c_3 + c_2 - c_1}{2c_2} + 1 \right) \frac{(c_2 - c_1)(c_3 - c_2)}{c_1 + c_3 - c_2} + \frac{2 \sum_{i=1}^{3}(c_1 + c_2 + c_3 - 2c_i)}{(c_3 - c_2)^2} > 0$$

Therefore, (38) is increasing in κ. Hence, there exists λ such that the lemma holds. ∎

Proof of Proposition 7. Consider $c_3 > c_1 + c_2$ first. Then, $c_1/(c_3 - c_2) < 1$ and $2c_1/(c_3 - c_2 + c_1) < 1$. Therefore, Proposition 3 implies that the equilibrium is of Case I for $\Delta_1/\Delta_2 \geq 2c_1/(c_3 - c_2 + c_1)$. If $\Delta_1/\Delta_2 \in (0, 2c_1/(c_3 - c_2 + c_1))$, Proposition 4 implies that the equilibrium is of Case III. If $\Delta_1/\Delta_2 = 0$, $v_1 = v_2 > 0$, so the equilibrium is of Case IIa.

Consider $c_3 \leq c_1 + c_2$. First, Proposition 4 implies that the equilibrium is of Case IIa for $\Delta_1/\Delta_2 \leq 1$. Second, Proposition 3 implies that the equilibrium is of Case I for $\Delta_1/\Delta_2 \geq c_1/(c_3 - c_2)$. Third, according to Proposition 4, the equilibrium is of Case IIa if $1 < \Delta_1/\Delta_2 < c_1/(c_3 - c_2)$ and if $F_3'(\underline{s}_1^*) \geq 0$. Lemma 8 implies that the equilibrium is of Case IIa for $\Delta_1/\Delta_2 \in (1, \lambda]$. Then, the first step implies that the equilibrium is of Case IIa for $\Delta_1/\Delta_2 \leq \lambda$. Fourth, if $\Delta_1/\Delta_2 \in (\lambda, c_1/(c_3 - c_2))$, Lemma 8 implies that $1 < \Delta_1/\Delta_2 < c_1/(c_3 - c_2)$ and $F_3'(\underline{s}_1^*) < 0$. Then, Proposition 6 implies that the equilibrium is of Case IIb. ∎

Proof of Proposition 8. Lemma 2 implies that players $4, ..., n$ choose $s = 0$ with probability 1. In addition, Lemma 5 implies that they do not win any prize in any equilibrium. Therefore, their equilibrium payoffs are zero.

Next, consider players 1,2 and 3. Because players $4, ..., n$ do not choose positive bids and do not win any prize, players $1, 2, 3$'s strategies in every equilibrium in the n-player contest are also an equilibrium in the three-player contest. Then, the unique equilibrium of the three-player contest implies that the n-player contest has a unique equilibrium, in which players $1, 2, 3$ use the same strategies as in the three-player contest. Moreover, their payoffs in the n-player contest are also the same as those in the three-player contest. ∎

Calculation Details in Example 1 Substituting the values of c_i and v_k into (1) and (2), we obtain $G_1^2(s) = 4s$ and $G_2^2(s) = 3/4 + s$. Suppose players 1 and 2 use these strategies, player 3's payoff by choosing s is given by (3) and can be rewritten as $U_3(s|G_1^2, G_2^2) = -8s^2 + 2s + 9/4$, which is maximized by a unique best response $\hat{s}_3 = 1/8$, and the corresponding payoff is $\hat{u}_3 = 19/8$. Then, substituting $\hat{s}_3 = 1/8$ into (7) and (8), we have $x_1 = 31/8$ and $x_2 = 7/2$. Substituting $\hat{s}_3 = 1/8$ and $\hat{u}_3 = 19/8$ into (6), we obtain $x_3 = 13/4$. Following Definition ii), the threshold T satisfies $x_3 - c_3 T = 0$, so $T = 13/28$. According to Definition iii), player i's power is $w_i = x_i - c_i T$, so $w_1 = 191/56$, $w_2 = 23/14$ and $w_3 = 0$. Then, Proposition 1 implies $u_i^* = x_i$, so $u_1^* = 191/56$, $u_2^* = 23/14$ and $u_3^* = 0$. Recall that $\underline{s}_1^*$ is the smallest $s \geq 0$ satisfying (12). Substituting the values of u_i^*, v_k, c_i into (12), we can rewrite it as $-2 \left(\frac{7s}{3} \frac{23/14 + 4s}{3} \right) + 3 \left(\frac{7s}{3} + \frac{23/14 + 4s}{3} \right) - s = \frac{191}{56}$, whose solutions are $(67 \pm 5\sqrt{37})/112$. Therefore, $\underline{s}_1^* = (67 - 5\sqrt{37})/112$. According to Proposition 5, $\bar{s}_1^* = (v_1 - u_1^*)/c_1 = 33/56$ and $\bar{s}_3^* = T = 13/28$. Therefore, the supports are $[0, 13/28]$ for player 3, $[0, 33/56]$ for player 2, and $[(67 - 5\sqrt{37})/112, 33/56]$ for player 1.